Iterative Computation of Rate-Distortion Bounds for Scalable Source Coding

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We consider N-layer scalable source coding of a finite memoryless source $X \sim p_x$. Let $\mathbf{X}_i$ denote $X_1, \ldots, X_i$, where X_i is the reproduction at the ith layer. From [5], we know that a scalable coder can achieve the sequence of decreasing distortions $\mathbf{D} = \{D_i\}_{i=1}^N$ and increasing rates $\mathbf{R} = \{R_i\}_{i=1}^N$, if and only if there exists a conditional distribution $Q_{\mathbf{x}_N|x}$ such that

$$\begin{aligned} E\langle d(X, X_i)\rangle &\leq D_i \quad i = 1, \ldots, N \\ I(X; \mathbf{X}_i) &\leq R_i \quad i = 1, \ldots, N \,. \end{aligned}$$

The $2N$-dimensional achievability region $\boldsymbol{\mathcal{A}}$ is convex. Hence, in order to find a point on the boundary of $\boldsymbol{\mathcal{A}}$ with an inward normal vector ($\boldsymbol{\alpha} = \{\alpha_i\}_{i=1}^N, \boldsymbol{\beta} = \{\beta_i\}_{i=1}^N$), we must solve the following minimization problem:

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \inf_{Q_{\mathbf{x}_N|x}} \sum_{i=1}^N \alpha_i I(X; \mathbf{X}_i) + \beta_i E\langle d(X, X_i)\rangle \,.$$

The above problem was first addressed by Effros [4, Section V]. A new system of equations and inequalities regarding the optimal marginal $q_{\mathbf{x}_N}$ was formed, and all tentative solutions (extracted from the equations) were tried until the one satisfying the optimality conditions (the inequalities in the system) was found. (See [1, Section 2.6] for the details of the approach for the ordinary rate-distortion problem.) However, it was not clear how $q_{x_{i+1}|\mathbf{x}_i}$ should be defined when $q_{\mathbf{x}_i} = 0$. In fact, we showed that satisfaction of the conditions given in [4] for some assumed $q_{x_{i+1}|\mathbf{x}_i}$ when $q_{\mathbf{x}_i} = 0$, does not necessarily imply the optimality of $q_{\mathbf{x}_N}$. Moreover, this approach becomes impractical as the size of the output alphabet grows. (For an extreme example, consider continuous source and reproduction alphabets.) As a remedy, we propose an iterative algorithm which is a generalization of the Blahut-Arimoto (BA) algorithm [2] for rate-distortion computation. The algorithm is initialized with arbitrary nonzero reproduction probabilities, and monotonically approaches the optimal reproduction distribution. We also revise the optimality conditions to handle the complications that arise whenever $q_{\mathbf{x}_i} = 0$.

Let $\mathbf{Q} = \{Q_{\mathbf{x}_N|x}\}$ and $\mathbf{q} = \{q_{\mathbf{x}_N}\}$ denote, in vector notation, a random encoding, and a reproduction distribution, respectively.

Lemma 1:

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \inf_{\mathbf{Q}} \inf_{\mathbf{q}} F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbf{Q}, \mathbf{q}) \,,$$

where

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbf{Q}, \mathbf{q}) \triangleq \sum_{i=1}^N \beta_i E_{\mathbf{Q}}\langle d(X, X_i)\rangle + \alpha_i \mathcal{D}(Q_{\mathbf{x}_i|x} p_x \| q_{\mathbf{x}_i} p_x)$$

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Thus, the problem is that of double minimization and, as will be shown, is solvable by alternating minimization.

Lemma 2:

a) Given $\mathbf{Q}$, $\arg\inf_{\mathbf{q}} F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbf{Q}, \mathbf{q})$ is the marginal

$$q_{\mathbf{x}_N}(\mathbf{Q}) = \sum_x p_x Q_{\mathbf{x}_N|x}$$

b) Given $\mathbf{q}$, $\arg\inf_{\mathbf{Q}} F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbf{Q}, \mathbf{q})$ is given by

$$Q_{\mathbf{x}_N|x}(\mathbf{q}) = \frac{q_{\mathbf{x}_N} \exp\{-\sum_{i=1}^N \beta_i' d_{x,x_i} + \alpha_i' \log f^i_{x,\mathbf{x}_i}\}}{\sum_{\mathbf{z}_N} q_{\mathbf{z}_N} \exp\{-\sum_{i=1}^N \beta_i' d_{x,z_i} + \alpha_i' \log f^i_{x,\mathbf{z}_i}\}} \,,$$

where $\alpha_i' = \alpha_i / \sum_{j=i}^N \alpha_j$ and $\beta_i' = \beta_i / \sum_{j=i}^N \alpha_j$, and

$$f^i_{x,\mathbf{x}_i} = \sum_{z_{i+1}} q_{z_{i+1}|\mathbf{x}_i} \exp\{-\beta_{i+1}' d_{x,z_{i+1}}\} (f^{i+1}_{x,\mathbf{x}_i,z_{i+1}})^{1-\alpha_{i+1}} \,,$$

for $i = 1, \ldots, N-1$, and $f^N_{x,\mathbf{x}_N} = 1$.

Theorem 1: Let $\mathbf{q}^{(0)}$ be positive everywhere, and let $\mathbf{Q}^{(n)} = \mathbf{Q}(\mathbf{q}^{(n-1)})$, and $\mathbf{q}^{(n)} = \mathbf{q}(\mathbf{Q}^{(n)})$ for $n = 1, 2, 3, \ldots$. Then the sequence $\mathbf{q}^{(0)}$, $\mathbf{Q}^{(1)}$, $\mathbf{q}^{(1)}$, $\mathbf{Q}^{(2)}$, ..., converges to

$$(\mathbf{Q}^*, \mathbf{q}^*) = \arg\inf_{\mathbf{Q},\mathbf{q}} (F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbf{Q}, \mathbf{q})) \,.$$

The proof follows the same line as the proof for the optimality of BA, given in [3]. Finally, the optimality conditions are given by

Theorem 2: A given $\mathbf{q}$ is optimal if and only if there exists a legitimate $q_{x_{i+1}|\mathbf{x}_i}$ for all $q_{\mathbf{x}_i} = 0$, so that

$$v_{\mathbf{x}_N} \leq v_{\mathbf{x}_{N-1}} \leq \cdots \leq v_{x_1} \leq 1 \,,$$

for all $\mathbf{x}_N$, where

$$v_{\mathbf{x}_j} = \sum_x \frac{p_x f^j_{x,\mathbf{x}_j} \exp\{-\sum_{i=1}^j \beta_i' d_{x,x_i} + \alpha_i' \log f^i_{x,\mathbf{x}_i}\}}{\sum_{\mathbf{z}_N} q_{\mathbf{z}_N} \exp\{-\sum_{i=1}^N \beta_i' d_{x,z_i} + \alpha_i' \log f^i_{x,\mathbf{z}_i}\}} \,.$$

References

[1] T. Berger. *Rate Distortion Theory. A Mathematical Basis for Data Compression.* Prentice-Hall, Englewood Cliffs, N.J., 1971.

[2] R. E. Blahut. Computation of channel capacity and rate-distortion functions. *IEEE Trans. on Information Theory*, 18(4):460–473, July 1972.

[3] I. Csiszár. On the computation of rate-distortion functions. *IEEE Trans. on Information Theory*, 20(1):122–124, January 1974.

[4] M. Effros. Distortion-rate bounds for fixed- and variable-rate multiresolution source coding. *IEEE Trans. on Information Theory*, 45(6):1887–1910, September 1999.

[5] B. Rimoldi. Successive refinement of information: Characterization of the achievable rates. *IEEE Trans. on Information Theory*, 40(1):253–259, January 1994.