# On Zero-Error Coding of Correlated Sources

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***Abstract*—The problem of separate *zero-error* coding of correlated sources is considered. Inner and outer single-letter bounds are established for the achievable rate region, and conditions for their coincidence are investigated. It is shown that successive encoding combined with time sharing is not always an optimal coding strategy. Conditions for its optimality are derived.**

**The inner bound to the achievable rate region follows as a special case of the single-letter characterization of a generalized zero-error multiterminal rate-distortion problem. The applications of this characterization to a problem of remote computing are also explored. Other results include i) a product-space characterization of the achievable rates, ii) bounds for finite block length, and iii) asymptotic fixed-length rates.**

***Index Terms*—Graph coloring, graph entropy, separate coding of correlated sources, Slepian–Wolf, zero-error information theory.**

## I. Introduction

Consider the multiterminal system shown in Fig. 1, where two correlated sources are encoded separately and decoded jointly while no communication is permitted between the encoders. We study the rates of transmission when the receiver is required to reproduce the source signals *without any error* (i.e., with zero error). We derive single-letter bounds for the asymptotically achievable rate region for both fixed- and variable-length codes. While the outer bound for variable-length coding is based on the results of Slepian and Wolf in [22], the inner bound follows from the exact characterization of a generalized zero-error multiterminal rate-distortion problem. We also derive bounds for achievable rates for a finite block length.

The problem of determining the asymptotically achievable rate region when the receiver is required to reproduce the sources *with a vanishingly small probability of error* was completely solved by Slepian and Wolf in their classic paper [22]. On the other hand, for the zero-error version of the problem, a computable characterization is currently not known even for the special point-to-point case, where the receiver has correlated side information unknown to the sender (i.e., the "unknown side information problem"). In fact, we have shown in [17] that a formula for the minimum rate of transmission in the latter problem would easily yield a solution for the famous open problem of determination of the zero-error capacity [20] of an arbitrary discrete memoryless channel.[1]

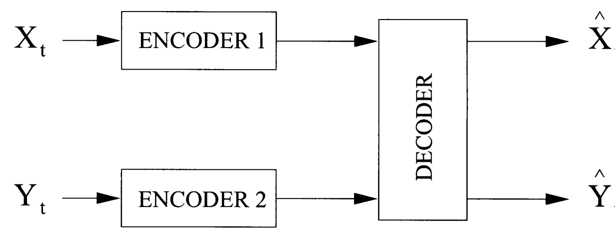


Fig. 1. Separate coding of correlated sources.

Our inner bound is defined as a functional on a graph associated with the problem, and represents an extension of the concept of *graph entropy*, an information-theoretic functional on graphs introduced by Körner in [12]. Graph entropy was used in [1] to give an upper bound to the minimum asymptotic rate for the unknown side information problem. This functional has also been applied in purely graph-theoretic problems, such as characterization of normal graphs and perfect graphs, and to derive lower bounds for perfect hashing and Boolean circuit sizes. (See [21] for a survey of some of these applications.) The inner bound derived in this paper also seems to be of independent interest; we use its generalization to determine the achievable rate region for a problem of remote computing, thus, partially extending the results of [19].

The practical aspects of the coding of correlated sources have recently received attention in the context of the spread of multiterminal networks. For example, consider remote low-power sensors which sense correlated versions of the same physical phenomenon, and separately convey their measurements to a central processor. In low-delay applications, the remote sensors may need to use zero-error codes rather than codes with asymptotically vanishing error (in the sense of Slepian–Wolf [22]). Motivated by such applications, the design of zero-error codes was previously considered in [11], [24], [25]. For a survey of results on restricted versions of the problem, and on variants where interaction is allowed, see [16]. Also see [10] for an earlier approach.

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[1] It was known from [18] that a formula for the *complementary graph entropy* of an arbitrary graph immediately yields a formula for the zero-error capacity. We have shown in [17] that the minimum achievable rate in the unknown side information problem is in fact given by the complementary graph entropy.

This paper is organized as follows. In the next section, we formally define different classes of zero-error codes, and demonstrate the differences on a running example, and provide related definitions. In Section III, we analyze the asymptotically achievable rate region for variable-length coding. After deriving a product-space characterization, we provide single-letter inner and outer bounds. We also show that our derivations specialize to known results for the unknown side information problem. Section IV is devoted to the derivation of conditions for the tightness of the inner and outer bounds. In Sections V and VI, bounds are obtained for the achievable rate regions for variable-length coding with finite block length, and for fixed-length coding with infinite block length, respectively. We conclude with a short section summarizing the results.

## II. Preliminaries and Notation

Let the pair of memoryless correlated sources $(\mathcal{S}_X, \mathcal{S}_Y)$ produce, at each instant, a pair of letters $(x, y)$ from the product set $\mathcal{X} \times \mathcal{Y}$ according to the joint probability distribution $P_{XY}(x, y)$. Let $P_X(x)$ and $P_Y(y)$ denote the corresponding marginal distributions over $\mathcal{X}$ and $\mathcal{Y}$, respectively. We may assume without loss of generality that $P_X(x) > 0\ \forall x \in \mathcal{X}$, and $P_Y(y) > 0\ \forall y \in \mathcal{Y}$. Let $x_m^n = (x_m, x_{m+1}, \ldots, x_n)$ and $y_m^n = (y_m, y_{m+1}, \ldots, y_n)$, for $1 \le m \le n$. Since $(\mathcal{S}_X, \mathcal{S}_Y)$ is memoryless, the probability of occurrence of the pair $(x_1^n, y_1^n) \in \mathcal{X}^n \times \mathcal{Y}^n$ for $n \ge 1$ is

$$P_{XY}^n(x_1^n, y_1^n) = \prod_{i=1}^{n} P_{XY}(x_i, y_i).$$

Similarly

$$P_X^n(x_1^n) = \prod_{i=1}^{n} P_X(x_i)$$

and

$$P_Y^n(y_1^n) = \prod_{i=1}^{n} P_Y(y_i)$$

are the marginal probabilities of $x_1^n$ and $y_1^n$, respectively.

Suppose that Alice has access to the source $\mathcal{S}_X$, while Bob has access to $\mathcal{S}_Y$, and they wish to convey their respective values to Merlin *without any error*. Both Alice and Bob know the underlying distribution $P_{XY}(x, y)$, but no communication is permitted between them. Suppose Alice and Bob use the *fixed-length* encoding functions

$$\phi_X : \mathcal{X}^n \longrightarrow \{1, 2, \ldots, N_X\}$$
$$\phi_Y : \mathcal{Y}^n \longrightarrow \{1, 2, \ldots, N_Y\}$$

respectively. Correspondingly, Merlin uses the decoding function

$$\psi : \{1, 2, \ldots, N_X\} \times \{1, 2, \ldots, N_Y\} \longrightarrow \mathcal{X}^n \times \mathcal{Y}^n.$$

In order for the triplet $(\phi_X, \phi_Y, \psi)$ to constitute a valid code, for each encoder output pair

$$(i, j) \in \{1, 2, \ldots, N_X\} \times \{1, 2, \ldots, N_Y\}$$

*at most one* source pair $(x_1^n, y_1^n) \in \phi_X^{-1}(i) \times \phi_Y^{-1}(j)$ must satisfy $P_{XY}^n(x_1^n, y_1^n) > 0$. If such a source pair $(x_1^n, y_1^n)$ exists, then obviously

$$\psi(i, j) = (x_1^n, y_1^n)$$

and if no such pair exists, then the encoder output pair $(i, j)$ can never occur, and the value of $\psi(i, j)$ is irrelevant. In Fig. 2, a valid fixed-length scalar $(n = 1)$ code is shown for an example distribution over $\mathcal{X} = \mathcal{Y} = \{\mathrm{a, b, c, d, e}\}$.

$P_{XY}(x, y)$:

| $x$ \ $y$ | a | b | c | d | e |
|---|---|---|---|---|---|
| a | 0.1 | 0.1 | 0 | 0 | 0 |
| b | 0 | 0.1 | 0.1 | 0 | 0 |
| c | 0 | 0 | 0.1 | 0.1 | 0 |
| d | 0 | 0 | 0 | 0.1 | 0.1 |
| e | 0.1 | 0 | 0 | 0 | 0.1 |

| $x$ | $\phi_X(x)$ |
|---|---|
| a | 1 |
| b | 2 |
| c | 1 |
| d | 2 |
| e | 3 |

| $y$ | $\phi_Y(y)$ |
|---|---|
| a | 1 |
| b | 2 |
| c | 3 |
| d | 4 |
| e | 5 |

$\psi(\phi_X(x), \phi_Y(y))$:

| $\phi_X(x)$ \ $\phi_Y(y)$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | (a,a) | (a,b) | (c,c) | (c,d) | (*,*) |
| 2 | (*,*) | (b,b) | (b,c) | (d,d) | (d,e) |
| 3 | (e,a) | (*,*) | (*,*) | (*,*) | (e,e) |

Fig. 2. The joint distribution $P_{XY}(x, y)$ (top) and a valid fixed-length coding triplet $(\phi_X, \phi_Y, \psi)$ (middle and bottom). The symbol $*$ stands for a "don't care" output.

A rate pair $(R_X, R_Y)$ is achievable by a fixed-length zero-error code if for any $\epsilon > 0$, there exists large enough $n$ such that

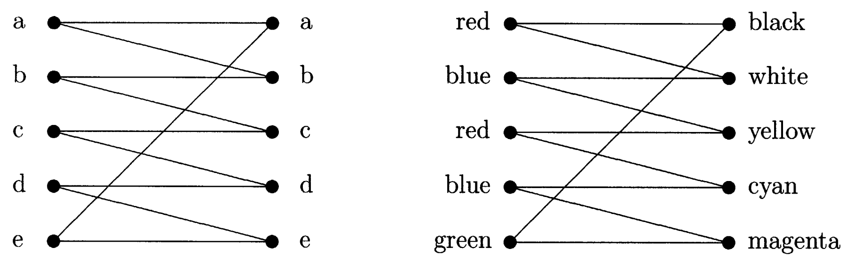
$$\frac{1}{n}\log N_X \le R_X + \epsilon$$
$$\frac{1}{n}\log N_Y \le R_Y + \epsilon$$

where here, and in the sequel, logarithms are taken to base 2. Hence, for the example in Fig. 2, the achievable rate region contains $(\log 3, \log 5)$.

The most natural and suitable setting for the problem of determining achievable rate pairs is, as will soon be clear, that of graph theory. Therefore, we proceed by providing necessary graph-theoretic definitions. Construct the bipartite graph $G = (\mathcal{X} \cup \mathcal{Y}, E)$ by setting, for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$

$$\{x, y\} \in E \Longleftrightarrow P_{XY}(x, y) > 0.$$

We call $G$ the *characteristic graph* of the correlated sources $(\mathcal{S}_X, \mathcal{S}_Y)$. Fig. 3(a) shows the characteristic graph corresponding to the joint distribution $P_{XY}(x, y)$ in Fig. 2. This particular graph is known as the Shannon typewriter channel.




(a) (b)

Fig. 3. The characteristic graph (a) and the bipartite coloring (b) corresponding to $P_{XY}$ and $(\phi_X, \phi_Y)$, respectively, in Fig. 2.

For $n \geq 1$, define the $n$-fold AND power of $G$ as the bipartite graph $G^n = (\mathcal{X}^n \cup \mathcal{Y}^n, E^n)$, with

$$\{x_1^n, y_1^n\} \in E^n \iff \{x_i, y_i\} \in E, \quad \text{for all } i = 1, \ldots, n \quad (1)$$

for $(x_1^n, y_1^n) \in \mathcal{X}^n \times \mathcal{Y}^n$. Thus $\{x_1^n, y_1^n\} \in E^n$ if and only if $P_{XY}^n(x_1^n, y_1^n) > 0$. The subgraph *induced* in $G$ by a subset of nodes $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$ is the graph $(\mathcal{X}' \cup \mathcal{Y}', E')$ with $\{x, y\} \in E'$ if and only if $\{x, y\} \in E$. A bipartite graph $G = (\mathcal{X} \cup \mathcal{Y}, E)$ is said to be *complete* if $\{x, y\} \in E$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

*Definition 1:* Assignments $\Phi_{\mathcal{X}}(\cdot)$ and $\Phi_{\mathcal{Y}}(\cdot)$ of "colors" (or labels), respectively, to the $\mathcal{X}$- and $\mathcal{Y}$-nodes of the bipartite graph $G = (\mathcal{X} \cup \mathcal{Y}, E)$ is called a *bipartite coloring* of $G$ if, for each color pair $(i, j)$, the subgraph induced by $\Phi_{\mathcal{X}}^{-1}(i) \times \Phi_{\mathcal{Y}}^{-1}(j)$ has at most one edge.

Observe the one-to-one correspondence between fixed-length zero-error codes and bipartite colorings of the graph $G^n$. Fig. 3(b) shows the bipartite coloring corresponding to the encoder pair $(\phi_X, \phi_Y)$ in Fig. 2. Note that the set of valid fixed-length codes, and hence, the set of rates achievable by such codes, are completely captured by $G$ and do not further depend on $P_{XY}$ ($G$ is completely determined by whether or not $P_{XY}(x, y) > 0$). We therefore denote the achievable rate region by $\mathcal{R}^{\text{fl}}(G)$.

It is more efficient in terms of expended rates to use *variable-length* coding, where Alice and Bob, respectively, use the encoding functions

$$\phi_X : \mathcal{X}^n \longrightarrow \{0, 1\}^*$$

and

$$\phi_Y : \mathcal{Y}^n \longrightarrow \{0, 1\}^*.$$

The encoded bitstream produced by Alice (Bob) is the concatenation of the consecutive outputs of $\phi_X$ ($\phi_Y$), i.e., Alice and Bob send

$$\phi_X(x_1^n)\phi_X(x_{n+1}^{2n})\phi_X(x_{2n+1}^{3n}) \cdots$$

and

$$\phi_Y(y_1^n)\phi_Y(y_{n+1}^{2n})\phi_Y(y_{2n+1}^{3n}) \cdots$$

respectively. Correspondingly, Merlin's decoding function, $\psi$, operates on the two encoded bitstreams and is required to output both source sequences without error. A rate pair $(R_X, R_Y)$ is achievable by a variable-length zero-error code if for any $\epsilon > 0$, there exists large enough $n$ such that

$$\frac{1}{n} \sum_{x_1^n \in \mathcal{X}^n} P_X^n(x_1^n) |\phi_X(x_1^n)| \leq R_X + \epsilon$$
$$\frac{1}{n} \sum_{y_1^n \in \mathcal{Y}^n} P_Y^n(y_1^n) |\phi_Y(y_1^n)| \leq R_Y + \epsilon.$$

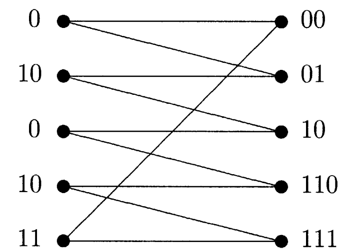



Fig. 4. A prefix-free code for the bipartite coloring in Fig. 3(b).

The simplest variable-length coder is obtained by designing a bipartite coloring $\Phi_{\mathcal{X}} \times \Phi_{\mathcal{Y}}$ of $G^n$, followed by encoding the two sets of colors (outputs of $\Phi_{\mathcal{X}}$ and $\Phi_{\mathcal{Y}}$) separately in a prefix-free fashion. The decoder first resolves the corresponding colors by separately parsing the bitstreams, and then decodes the unique edge of $G^n$ determined by the decoded colors. We refer to this scheme as *prefix-free coding*. Fig. 4 shows a prefix-free code corresponding to the bipartite coloring in Fig. 3(b). As in the case of fixed-length codes, the set of valid prefix-free codes is completely determined by $G$. The expected rates are functionals of only the marginals $P_X$ and $P_Y$, and do not require any further information provided by $P_{XY}$. Hence, the set of achievable rates is completely determined by $G$, $P_X$, and $P_Y$, and will be denoted by $\mathcal{R}^{\text{pf}}(G, P_X, P_Y)$.

More complicated variable-length coding schemes can be constructed. Let us first consider the *instantaneous codes* defined in [24].

*Definition 2:* The triplet $(\phi_X, \phi_Y, \psi)$ is an instantaneous zero-error code if the decoder $\psi$ reconstructs $(x_1^n, y_1^n)$ without any error by reading only the first $|\phi_X(x_1^n)|$ and $|\phi_Y(y_1^n)|$ bits from the two encoded bitstreams, respectively.

Note that the decoder does not know $|\phi_X(x_1^n)|$ or $|\phi_Y(y_1^n)|$ in advance. It was shown in [25] that $(\phi_X, \phi_Y, \psi)$ is an instantaneous code if and only if, for every distinct pair $(x_1^n, y_1^n), (x_1'^n, y_1'^n) \in \mathcal{X}^n \times \mathcal{Y}^n$

$$\{x_1^n, y_1^n\} \in E^n, \{x_1'^n, y_1'^n\} \in E^n \Longrightarrow (\phi_X(x_1^n), \phi_Y(y_1^n)) \neq_p (\phi_X(x_1'^n), \phi_Y(y_1'^n)). \quad (2)$$

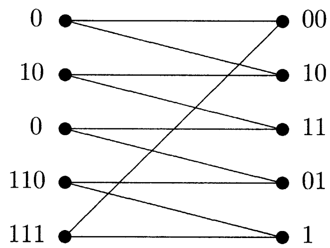



Fig. 5. An instantaneous code which is not prefix free. It is easy to check that (2) is indeed satisfied.

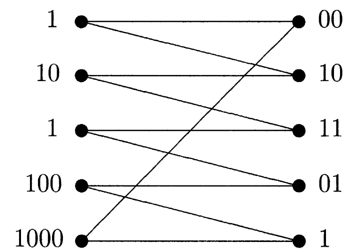



Fig. 6. A uniquely decodable code which is not instantaneous.

Here, $(a,b)\neq_p(a',b')$, when $a, a', b$ and $b'$ are binary strings, means that either $a$ is not a prefix of $a'$, or $b$ is not a prefix of $b'$, or both these conditions are satisfied. We show in Fig. 5 an instantaneous code for our running example, which is *not* prefix free (because both 1 and 11 are codewords assigned by $\phi_Y$). Observe that prefix-free codes automatically satisfy (2), and therefore, form a subclass of instantaneous codes. Also, the set of achievable rates is completely determined (cf.(2)) by $G$, $P_X$, and $P_Y$, and will be denoted by $\mathcal{R}^{\text{inst}}(G,P_X,P_Y)$.

The most general class of variable-length zero-error correlated source codes is that of *uniquely decodable codes*, which we define below. A uniquely decodable coding scheme which is not instantaneous has a high decoding complexity, as the decoder may have to buffer the entire pair of bitstreams before starting the decoding process.

*Definition 3:* The triplet $(\phi_X,\phi_Y,\psi)$ is a uniquely decodable zero-error code if for any $N \geq 1$, and for every finite sequence $(x_1^{Nn}, y_1^{Nn})$ with $P_{XY}^{Nn}(x_1^{Nn}, y_1^{Nn}) > 0$

$$\psi\Big(\phi_X(x_1^n)\phi_X(x_{n+1}^{2n})\cdots\phi_X(x_{Nn-n+1}^{Nn}),\ \phi_Y(y_1^n)\phi_Y(y_{n+1}^{2n})\cdots\phi_Y(y_{Nn-n+1}^{Nn})\Big) = (x_1^{Nn}, y_1^{Nn}). \quad (3)$$

Fig. 6 shows a uniquely decodable code which is *not* instantaneous, as the pairs $(b,c)$ and $(d,e)$ are, respectively, assigned codewords $(10,11)$ and $(100,1)$, i.e., (2) is violated. Yet, the code is uniquely decodable because the $\phi_X$-bitstream can itself be uniquely parsed (as it is a concatenation of codewords $1, 10, 100,$ and 1000), and once $\phi_X(x)$ is recovered, the codewords of all possible $y$ form a prefix-free code, thus allowing $\phi_Y(y)$ also to be recovered. Then it is easy to recover $\{x,y\}$, because $\phi_X$ and $\phi_Y$ induce a bipartite coloring of $G$.

Since the condition $P_{XY}^{Nn}(x_1^{Nn}, y_1^{Nn}) > 0$ in Definition 3 may be recast as

$$\{x_{kn-n+1}^{kn}, y_{kn-n+1}^{kn}\} \in E^n, \qquad \text{for all } k = 1,\ldots,N \quad (4)$$

the region of achievable rates is a function of only $G$, $P_X$, and $P_Y$, and will be denoted as $\mathcal{R}^{\text{ud}}(G,P_X,P_Y)$.

From the foregoing, it is clear that

$$\begin{aligned}\mathcal{R}^{\text{fl}}(G) &\subseteq \mathcal{R}^{\text{pf}}(G,P_X,P_Y)\\ &\subseteq \mathcal{R}^{\text{inst}}(G,P_X,P_Y)\\ &\subseteq \mathcal{R}^{\text{ud}}(G,P_X,P_Y)\end{aligned}$$

for all $G$, $P_X$, and $P_Y$. One might expect each of the above achievability regions to be strictly contained by the next. However, one of our main results in this paper is the following theorem.

*Theorem 1:*

$$\begin{aligned}\mathcal{R}^{\text{pf}}(G,P_X,P_Y) &= \mathcal{R}^{\text{inst}}(G,P_X,P_Y)\\ &= \mathcal{R}^{\text{ud}}(G,P_X,P_Y)\\ &\stackrel{\text{def}}{=} \mathcal{R}^{\text{vl}}(G,P_X,P_Y)\end{aligned} \quad (5)$$

that is, the larger class of uniquely decodable codes offer no *asymptotic* advantage over the simple prefix-free codes.

Therefore, for the purpose of characterizing the asymptotic rate region, we can treat prefix-free codes as the entire variable-length coding class. After proving this fact, we will derive an inner bound to $\mathcal{R}^{\text{vl}}(G,P_X,P_Y)$ by constructing actual prefix-free codes for large $n$. The derivation utilizes the concept of *bipartite covers*, i.e., a generalization of bipartite coloring.

*Definition 4:* Let $\mathcal{C} = (\mathcal{C}_X, \mathcal{C}_Y)$, where $\mathcal{C}_X$ and $\mathcal{C}_Y$ are (possibly overlapping) collections of subsets of $\mathcal{X}$ and $\mathcal{Y}$, respectively. We will call $\mathcal{C}$ a bipartite cover of $G$ if

1) every $x \in \mathcal{X}$ $(y \in \mathcal{Y})$ is contained in some $s \in \mathcal{C}_X$ $(t \in \mathcal{C}_Y)$ and
2) for every $s \in \mathcal{C}_X$ and $t \in \mathcal{C}_Y$, the set $s \cup t$ induces a subgraph in $G$ with *at most one edge* (from $E$).

Note that each pair $(s,t)$, with $s \in \mathcal{C}_X$ and $t \in \mathcal{C}_Y$, identify a unique edge of $G$. Also, the color classes in any bipartite coloring for $G$ induce a bipartite cover, where color classes play the role of subsets in $\mathcal{C}_X$ and $\mathcal{C}_Y$. The only difference is that the subsets in $\mathcal{C}_X$ (or $\mathcal{C}_Y$) of a bipartite cover can be overlapping in general.

We will call the bipartite cover $\mathcal{C} = (\mathcal{C}_X, \mathcal{C}_Y)$ an *exact bipartite cover* if, for every $s \in \mathcal{C}_X$ and $t \in \mathcal{C}_Y$, the set $s \cup t$ induces a subgraph in $G$ with *exactly one edge* (from $E$). Note that every bipartite graph $G = (\mathcal{X} \cup \mathcal{Y}, E)$ has a bipartite cover: Take $\mathcal{C}_X = \{\{x\} : x \in \mathcal{X}\}$ and $\mathcal{C}_Y = \{\{y\} : y \in \mathcal{Y}\}$. But not every bipartite graph has an exact bipartite cover. We shall need this important fact in the sequel. For example, consider the "Z-shaped" graph

$$\begin{aligned}G &= (\mathcal{X} \cup \mathcal{Y}, E)\\ &= (\{x_1,x_2,y_1,y_2\}, \{\{x_1,y_1\},\{x_2,y_1\},\{x_2,y_2\}\}).\end{aligned}$$

The only bipartite cover for this graph is $\mathcal{C}_X = \{\{x_1\},\{x_2\}\}$ and $\mathcal{C}_Y = \{\{y_1\},\{y_2\}\}$, which is not exact.

Often of interest is the minimum achievable rate in a restricted correlated source coding scenario, which is commonly referred to as the "unknown side information problem." In this scheme, Bob directly encodes his information $y_1^n$ without any

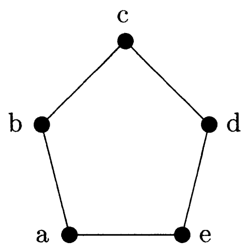



Fig. 7. The characteristic graph $G_X$ for successive coding of the source in Fig. 3(a).

error, expending asymptotic rate $H(Y)$. Merlin then recovers $y_1^n$ and Alice encodes $x_1^n$ while treating $y_1^n$ as the unknown side information available to Merlin. The following graphical construction was introduced by Witsenhausen in [23] to tackle this scheme which we call *successive encoding*: Form the bipartite graph $G = (\mathcal{X} \cup \mathcal{Y}, E)$ and define the graph $G_X = (\mathcal{X}, E_X)$ on the vertex set $\mathcal{X}$ by connecting distinct $x$ and $x'$ if there exists $y \in \mathcal{Y}$ such that $\{x, y\} \in E$ and $\{x', y\} \in E$. See Fig. 7 for the graph $G_X$ corresponding to $G$ in Fig. 3(a). The $n$-fold AND power $G_X^n = (\mathcal{X}^n, E_X^n)$ is defined in a manner analogous to (1): for distinct $x_1^n, x_1'^n \in \mathcal{X}^n$

$$\{x_1^n, x_1'^n\} \in E_X^n \iff \text{either } x_i = x_i' \text{ or } \{x_i, x_i'\} \in E_X \text{ for } 1 \le i \le n.$$

One can change the roles of $\mathcal{X}$ and $\mathcal{Y}$, and define $G_Y = (\mathcal{Y}, E_Y)$ and $G_Y^n$ similarly. A valid *coloring* of $G_X$ is a mapping from nodes to colors such that no two nodes connected with an edge are assigned the same color. Now, there is a one-to-one correspondence between fixed-length codes that Alice can use and colorings of graph $G_X$. Classes of variable-length codes are defined similarly to the general case. We proved in [17] using a result of Alon and Orlitsky [1] that for the determination of the minimum asymptotic rate Alice can achieve, it suffices to focus on the variable-length codes which simply encode a valid graph coloring in a prefix-free manner. That result, in fact, follows easily from Theorem 1.

Let us conclude this section by outlining some notations and constructions defined on the graphs $G$, $G_X$, and $G_Y$ that will be needed in the sequel.

Let $F = (V_F, E_F)$ be an arbitrary graph with no loops or multiple edges. Two vertices are connected in $\overline{F}$—the complement of $F$—if they are not connected in $F$. A subgraph induced by the vertex set $U_F \subseteq V_F$ is said to be *complete* if $\{v, v'\} \in E_F$ for every distinct pair $v, v' \in U_F$. A complete subgraph is also said to be *maximal* if it is not induced in any other complete subgraph in $F$. Denote by $\mathcal{T}(F)$ the collection of all maximal complete subgraphs of $F$. Note that the corresponding collection of all maximal complete subgraphs in $F^n$ is the $n$-fold Cartesian power $\mathcal{T}(F)^n$. (See [3, Ch. 16, Proposition 8] for a proof of this fact.) Thus, denoting by $\omega(F)$ the size of the largest maximal complete subgraph of $F$, we have that $\omega(F^n) = \omega(F)^n$. Also denote by $\chi(F)$ the minimum number of colors needed in a valid coloring of $F$. It immediately follows that $\chi(F) \ge \omega(F)$.

Given a bipartite cover $\mathcal{C} = (\mathcal{C}_X, \mathcal{C}_Y)$ of $G$, we will often define the random variables $S$ and $T$ via the collections of conditional probability distributions $p(s|x)$ and $p(t|y)$ of the following form.

1) $s$ and $t$ take values in $\mathcal{C}_X$ and $\mathcal{C}_Y$, respectively.
2) For each $x$, $p(s|x) > 0$ only if $x \in s$. Similarly, $p(t|y) > 0$ only if $y \in t$.

We will succinctly represent these conditions by employing the streamlined notation $X \in S \in \mathcal{C}_X$ and $Y \in T \in \mathcal{C}_Y$.

We will also need random variables $X'$ and $Y'$ jointly distributed according to $P(x, y)$ (not necessarily equal to $P_{XY}(x, y)$) on $\mathcal{X} \times \mathcal{Y}$, where $P(x, y)$ satisfies, for all $(x, y)$: $P(x, y) > 0$ only if $\{x, y\} \in E$, and

$$\sum_{y' \in \mathcal{Y}} P(x, y') = P_X(x)$$
$$\sum_{x' \in \mathcal{X}} P(x', y) = P_Y(y).$$

We will indicate such random variables by writing $\{X', Y'\} \in E$, and $X' \sim P_X$, $Y' \sim P_Y$. When we consider a random variable $S_X$ with conditional distribution $p(s_X|x)$ taking values $s_X \in \mathcal{T}(G_X)$ such that $p(s_X|x) > 0$ only if $x \in s_X$, we will denote it as $X \in S_X \in \mathcal{T}(G_X)$. A random variable $T_Y$ with $Y \in T_Y \in \mathcal{T}(G_Y)$ is interpreted similarly.

Sometimes we fix $G$ and consider the class of all marginals on $\mathcal{X}$ and $\mathcal{Y}$ obtained via joint distributions on $\mathcal{X} \times \mathcal{Y}$ whose characteristic graph is $G$. If the distributions $(P_X, P_Y)$ may be obtained as marginals of some joint distribution $P(x, y)$ whose characteristic graph is $G$ (i.e., $P(x, y) > 0$ if and only if $\{x, y\} \in E$), we will say that "$P_X$ and $P_Y$ are marginals on $G$".

## III. Asymptotic Bounds for Variable-Length Coding

We begin by proving Theorem 1. Toward this end, we first observe using standard time-sharing arguments that the rate region $\mathcal{R}^{\mathrm{pf}}(G, P_X, P_Y)$ achievable by prefix-free codes is convex. Therefore, its boundary can be characterized by the Lagrangian

$$L(G, P_X, P_Y, \alpha) \stackrel{\mathrm{def}}{=} \min_{(R_X, R_Y) \in \mathcal{R}^{\mathrm{pf}}(G, P_X, P_Y)} \left\{ \alpha R_X + (1 - \alpha) R_Y \right\}$$

evaluated for all $0 \le \alpha \le 1$. Motivated by this Lagrangian formulation, we define the *bipartite chromatic entropy* of $G$, which is an extension of the *chromatic entropy* defined in [1]. We then use this functional for a product-space characterization of $L(G, P_X, P_Y, \alpha)$.

*Definition 5:* The bipartite chromatic entropy of $G$ is given by

$$H_\chi(G, P_X, P_Y, \alpha) \stackrel{\mathrm{def}}{=} \min_{\Phi_X, \Phi_Y} \left\{ \alpha H(\Phi_X(X)) + (1 - \alpha) H(\Phi_Y(Y)) \right\} \quad (6)$$

where the minimization is with respect to all bipartite colorings $(\Phi_X, \Phi_Y)$ of $G$.

*Lemma 1:*

$$L(G, P_X, P_Y, \alpha) = \lim_{n \to \infty} \frac{1}{n} H_\chi(G^n, P_X^n, P_Y^n, \alpha). \quad (7)$$

*Proof:* The limit in (7) exists, since the chromatic entropy $H_\chi(G^n, P_X^n, P_Y^n, \alpha)$ is subadditive with respect to $n$, i.e.,

$$H_\chi(G^{n+m}, P_X^{n+m}, P_Y^{n+m}, \alpha) \leq H_\chi(G^n, P_X^n, P_Y^n, \alpha) + H_\chi(G^m, P_X^m, P_Y^m, \alpha). \quad (8)$$

To see this, let $(\Phi_X^{(n)}, \Phi_Y^{(n)})$ and $(\Phi_X^{(m)}, \Phi_Y^{(m)})$ be the bipartite colorings of $G^n$ and $G^m$, achieving $H_\chi(G^n, P_X^n, P_Y^n, \alpha)$ and $H_\chi(G^m, P_X^m, P_Y^m, \alpha)$, respectively. Define the functions $\Phi_X^{(n+m)}$ on $\mathcal{X}^{n+m}$, and $\Phi_Y^{(n+m)}$ on $\mathcal{Y}^{n+m}$, by

$$\Phi_X^{(n+m)}(x_1^{n+m}) = (\Phi_X^{(n)}(x_1^n), \Phi_X^{(m)}(x_{n+1}^{n+m}))$$
$$\Phi_Y^{(n+m)}(y_1^{n+m}) = (\Phi_Y^{(n)}(y_1^n), \Phi_Y^{(m)}(y_{n+1}^{n+m})).$$

Then (8) holds since $(\Phi_X^{(n+m)}, \Phi_Y^{(n+m)})$ is a bipartite coloring of $G^{n+m}$ achieving the right-hand side of (8).

Now, since every prefix-free code is obtained by combining a bipartite coloring with separate prefix-free coding of $\mathcal{X}$- and $\mathcal{Y}$-colors, it is clear that

$$\frac{1}{n}H_\chi(G^n, P_X^n, P_Y^n, \alpha) + \frac{1}{n} \geq L(G, P_X, P_Y, \alpha) \geq \frac{1}{n}H_\chi(G^n, P_X^n, P_Y^n, \alpha)$$

for any $n \geq 1$. The result follows by letting $n \to \infty$. □

*Proof (Theorem 1):* Consider a uniquely decodable code $(\phi_X, \phi_Y, \psi)$, operating with block length $n \geq 1$, and achieving $(R_X, R_Y)$. Set $N = 1$. It follows from (3) and (4) that

$$\{x_1^n, y_1^n\} \in E^n, \{x_1'^n, y_1'^n\} \in E^n \implies (\phi_X(x_1^n), \phi_Y(y_1^n)) \neq (\phi_X(x_1'^n), \phi_Y(y_1'^n))$$

since otherwise the decoder $\psi$ cannot distinguish $\{x_1^n, y_1^n\}$ and $\{x_1'^n, y_1'^n\}$. This, in turn, implies that $\phi_X \times \phi_Y$ induces a bipartite coloring, or in other words, $\phi_X$ and $\phi_Y$ are "one-to-one" codes for the $\mathcal{X}$- and the $\mathcal{Y}$-colors, respectively, in a bipartite coloring of $G^n$. Therefore, using Alon and Orlitsky's result on one-to-one codes [2], we can write

$$R_X \geq \frac{1}{n}\Big[H(\phi_X(X_1^n)) - \log\big(H(\phi_X(X_1^n))+1\big) - \log e\Big] \quad (9)$$

and similarly

$$R_Y \geq \frac{1}{n}\Big[H(\phi_Y(Y_1^n)) - \log\big(H(\phi_Y(Y_1^n))+1\big) - \log e\Big]. \quad (10)$$

Combining (9), (10), and (6) with the trivial observation that $H(\phi_X(X_1^n)) \leq n\log|\mathcal{X}|$ and $H(\phi_Y(Y_1^n)) \leq n\log|\mathcal{Y}|$, we obtain

$$\alpha R_X + (1-\alpha)R_Y \geq \frac{1}{n}H_\chi(G^n, P_X^n, P_Y^n, \alpha) - r(n) \quad (11)$$

for all $0 \leq \alpha \leq 1$, where

$$r(n) \stackrel{\text{def}}{=} \frac{1}{n}\Big[\alpha\log\big(n\log|\mathcal{X}|+1\big) + (1-\alpha)\log\big(n\log|\mathcal{Y}|+1\big) + \log e\Big].$$

Taking the limit $n \to \infty$ in (11), we obtain

$$L(G, P_X, P_Y, \alpha) \leq \min_{(R_X,R_Y)\in\mathcal{R}^{\text{ud}}(G,P_X,P_Y)} \Big\{\alpha R_X + (1-\alpha)R_Y\Big\}.$$

The proof is completed by observing the reversed inequality, which is implied by the fact that

$$\mathcal{R}^{\text{pf}}(G, P_X, P_Y) \subseteq \mathcal{R}^{\text{ud}}(G, P_X, P_Y). \quad \square$$

*Corollary 1:*

$$L(G, P_X, P_Y, \alpha) = \min_{(R_X,R_Y)\in\mathcal{R}^{\text{vl}}(G,P_X,P_Y)} \Big\{\alpha R_X + (1-\alpha)R_Y\Big\}$$

i.e., (7) is a product-space characterization for the entire class of variable-length codes.

Although the characterization (7) is complete, it is not computable in general. As mentioned earlier, the task of Alice and Bob is to inform Merlin, without error, of the edges occuring in $G^n$. By (1), an edge occurs in $G^n$ if and only if edges in $G$ occur in *all* the corresponding $n$ successive coordinates. Roughly speaking, the inability to construct codes which optimally exploit this interdependence between the coordinates is the crux of the difficulty in evaluating the achievable rate region $\mathcal{R}^{\text{vl}}(G, P_X, P_Y)$. A natural approach to bounding this region, which sidesteps consideration of the interdependence, is to require Merlin to recover every edge in $G$ without error even when such edges occur in *any* of the $n$ successive coordinates. In Sections III-B–III-D, we will present a step-by-step development of the *exact* achievable rate region for this stronger requirement, thus obtaining the inner bound

$$\mathcal{R}^{\text{in}}(G, P_X, P_Y) \subseteq \mathcal{R}^{\text{vl}}(G, P_X, P_Y).$$

Recall that the Slepian–Wolf result [22] is a single-letter characterization of the entire region of achievable rates for the weaker requirement of vanishingly small probability of error (instead of zero error). In Section III-E, we exploit that characterization to obtain the outer bound

$$\mathcal{R}^{\text{out}}(G, P_X, P_Y) \supseteq \mathcal{R}^{\text{vl}}(G, P_X, P_Y).$$

We will demonstrate in Section III-F that these bounds can be tight at certain rates, and that successive encoding followed by time sharing can be a suboptimal zero-error coding strategy, in contrast with the Slepian–Wolf setup.

### *A. A Generalized Multiterminal Rate-Distortion Problem*

Let $\mathcal{Z}$ be a finite set, and $d: \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to [0, \infty)$ be a single-letter distortion measure. Extend the definition of $d$ to vectors by setting, for $x^n \in \mathcal{X}^n$, $y^n \in \mathcal{Y}^n$, and $z^n \in \mathcal{Z}^n$

$$d(x^n, y^n, z^n) = \frac{1}{n}\sum_{i=1}^{n} d(x_i, y_i, z_i). \quad (12)$$

Let $P(x, y)$ be an arbitrary distribution on $\mathcal{X} \times \mathcal{Y}$ with

$$P(x, y) > 0, \qquad \text{for all } (x, y) \in \mathcal{X} \times \mathcal{Y}$$

and such that the corresponding marginals are $P_X$ and $P_Y$, respectively. Let the independent and identically distributed (i.i.d.) random variables $(X_1, Y_1), (X_2, Y_2), \ldots,$ be drawn from $P(x, y)$. We are interested in the rate region for separate encoding of $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ such that a joint decoder can estimate every realization $(X_i, Y_i)$ with zero distortion. More precisely, for the triplet $(\phi_X, \phi_Y, \psi)$ with $\phi_X : \mathcal{X}^n \longrightarrow \{0,1\}^*$, $\phi_Y : \mathcal{Y}^n \longrightarrow \{0,1\}^*$, and $\psi : \{0,1\}^* \times \{0,1\}^* \longrightarrow \mathcal{Z}^n$, where the codewords assigned by $\phi_X$ (and $\phi_Y$) do not prefix each other, we require

$$d(x_1^n, y_1^n, \psi(\phi_X(x_1^n), \phi_Y(y_1^n))) = 0 \tag{13}$$

for every finite sequence $(x_1^n, y_1^n)$. We denote the region of achievable rate pairs as $\mathcal{R}_d(P_X, P_Y)$, since the set of $(\phi_X, \phi_Y, \psi)$ satisfying (13) is fully captured by $d$, and the achieved expected rates

$$\frac{1}{n} \sum_{x_1^n \in \mathcal{X}^n} P_X^n(x_1^n) |\phi_X(x_1^n)|$$

and

$$\frac{1}{n} \sum_{y_1^n \in \mathcal{Y}^n} P_Y^n(y_1^n) |\phi_Y(y_1^n)|$$

are functionals of only the respective marginals $P_X$ and $P_Y$. In other words, there is no further dependence on $P(x, y)$.

Special cases of this multiterminal scenario were introduced earlier. For example, the multiterminal rate-distortion problem of [5] corresponds to $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ with $d(x, y, z) = 0$ if and only if $d_1(x, \hat{x}) = 0$ and $d_2(y, \hat{y}) = 0$. Another special case is introduced in [6] where again $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, and $d(x, y, z) = 0$ if and only if $x = \hat{x}$ and $d_2(y, \hat{y}) = 0$. Note that the condition (13) of exact reproduction for *every* $(x_1^n, y_1^n)$ is stronger than requiring exact reproduction only for those $(x_1^n, y_1^n)$ which are contained in some high-probability subset of $\mathcal{X}^n \times \mathcal{Y}^n$. The latter requirement leads to the multiterminal rate-distortion problem of [5], which remains unresolved at present. On the other hand, we derive below an exact single-letter characterization of $\mathcal{R}_d(P_X, P_Y)$ by showing that a relatively simple encoding strategy is already optimal for the stricter condition (13).

*Theorem 2:* $(R_X, R_Y) \in \mathcal{R}_d(P_X, P_Y)$ if and only if there exist: i) random variables $\Sigma, \Gamma$, and $Q$ jointly distributed with $X$ and $Y$ as $p(x, y, q, \sigma, \gamma) = p(q) p(\sigma|x, q) p(\gamma|y, q) P_X(x) P_Y(y)$ such that

$$R_X \geq I(X; \Sigma|Q) \quad \text{and} \quad R_Y \geq I(Y; \Gamma|Q) \tag{14}$$

and ii) functions $z_q(\sigma, \gamma)$ taking values in $\mathcal{Z}$ such that

$$\sum_q p(q) \sum_{x,y,\sigma,\gamma} P_X(x) P_Y(y) \cdot p(\sigma|x, q) p(\gamma|y, q) d(x, y, z_q(\sigma, \gamma)) = 0. \tag{15}$$

*Remarks:*

1) The sum in (15) consists of nonnegative terms, which must therefore all vanish. Since $P_X(x) P_Y(y) > 0$ for every pair $(x, y)$, we obtain that (15) requires

$$p(\sigma|x, q) p(\gamma|y, q) d(x, y, z_q(\sigma, \gamma)) = 0, \qquad \forall x, y, \sigma, \gamma \tag{16}$$

for every $q$ such that $p(q) > 0$.

2) $\mathcal{R}_d(P_X, P_Y)$ is convex, with $Q$ playing the role of the time-sharing random variable. It is known from Carathéodory's theorem [7, Theorem 14.3.4] that any point in the convex closure of a connected compact set in a $k$-dimensional Euclidean space can be represented as a convex combination of $k+1$ or fewer points in the same set. Therefore, for the computation of $\mathcal{R}_d(P_X, P_Y)$, we may assume that $Q$ takes values in $\{1, 2, 3\}$.

*Proof:*

*Direct:* Let the distributions $p(\sigma|x)$, $p(\gamma|y)$, and function $z(\sigma, \gamma)$ satisfying

$$\sum_{x,y,\sigma,\gamma} P_X(x) P_Y(y) p(\sigma|x) p(\gamma|y) d(x, y, z(\sigma, \gamma)) = 0 \tag{17}$$

be given. We will construct a sequence of prefix-free codes satisfying (13) whose corresponding rate pairs converge to $(I(X; \Sigma), I(Y; \Gamma))$. The forward part of the theorem will then follow from standard time-sharing arguments.

We will denote by $P_{X\Sigma}$ and $P_{Y\Gamma}$ the joint distributions of $X$ and $\Sigma$, and of $Y$ and $\Gamma$, respectively, and by $P_\Sigma(\sigma)$ and $P_\Gamma(\gamma)$ the marginal distributions $\sum_x P_X(x) p(\sigma|x)$ and $\sum_y P_Y(y) p(\gamma|y)$, respectively. Fix $\epsilon > 0$. Following the notation of [8], we denote by $T^n_{[P_X]_\epsilon}$ the set of sequences $x_1^n \in \mathcal{X}^n$ such that $N(a|x_1^n)$, the number of occurences of $a$ in $x_1^n$, satisfies

$$\left| \frac{1}{n} N(a|x_1^n) - P_X(a) \right| \leq \epsilon \text{ for every } a \in \mathcal{X}$$
$$\text{and no } a \in \mathcal{X} \text{with } P_X(a) = 0 \text{ occurs in } x_1^n \tag{18}$$

and call its members $(P_X, \epsilon)$-typical sequences. These may be simply referred to as $\epsilon$-typical sequences when the underlying distribution is clear from the context. Other typical sets and sequences in the subsequent are similarly defined.

By the type-covering lemma [8, Lemma 2.4.1], there exists $n_0(\epsilon)$ such that for every $n \geq n_0(\epsilon)$ there is a subset $\mathcal{C}_\Sigma$ of $T^n_{[P_\Sigma]_{2|\mathcal{X}|\epsilon}}$ with the following property: associated with every $x_1^n \in T^n_{[P_X]_\epsilon}$ is a $\sigma_1^n \in \mathcal{C}_\Sigma$ such that $(x_1^n, \sigma_1^n) \in T^n_{[P_{X\Sigma}]_{2\epsilon}}$. Similarly, there exists a subset $\mathcal{C}_\Gamma$ of $T^n_{[P_\Gamma]_{2|\mathcal{Y}|\epsilon}}$ which satisfies the corresponding property with respect to $T^n_{[P_Y]_\epsilon}$. Further, the cardinalities of $\mathcal{C}_\Sigma$ and $\mathcal{C}_\Gamma$ are not more than

$$2^{nI(X;\Sigma) + n\delta(\epsilon)} \quad \text{and} \quad 2^{nI(Y;\Gamma) + n\delta(\epsilon)}$$

respectively (where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$).

The encoding strategy is simple. If $x_1^n \in T^n_{[P_X]_\epsilon}$, Alice sends the index of $\sigma_1^n \in \mathcal{C}_\Sigma$ such that $(x_1^n, \sigma_1^n) \in T^n_{[P_{X\Sigma}]_{2\epsilon}}$. This step requires a rate of no more than $I(X; \Sigma) + \delta(\epsilon)$ bits. If $x_1^n \notin T^n_{[P_X]_\epsilon}$, she directly encodes it, expending no more than $\log|\mathcal{X}| + \frac{2}{n}$ bits. Bob similarly handles $y_1^n$. Now, for any $\epsilon > 0$, the probability that $x_1^n \notin T^n_{[P_X]_\epsilon}$ (correspondingly, $y_1^n \notin T^n_{[P_Y]_\epsilon}$) approaches 0 as $n \to \infty$, and it is easy to see that the rates $(I(X; \Sigma), I(Y; \Gamma))$ are then achieved.

Let us now turn to the decoder. When $x_1^n \in T^n_{[P_X]_\epsilon}$ and $y_1^n \in T^n_{[P_Y]_\epsilon}$, Merlin receives the indexes of $\sigma_1^n$ and $\gamma_1^n$, and reproduces

$$z^n(\sigma_1^n, \gamma_1^n) \stackrel{\text{def}}{=} (z(\sigma_1, \gamma_1), \ldots, z(\sigma_n, \gamma_n)).$$

But $(x_1^n, \sigma_1^n)$ and $(y_1^n, \gamma_1^n)$ are $2\epsilon$-typical. This implies, from the definition of typicality, that $P_X(x_i) p(\sigma_i|x_i) > 0$ and

$P_Y(y_i)p(\gamma_i|y_i) > 0$ for all $i = 1, \ldots, n$. It follows from (17) that

$$d(x_1^n, y_1^n, z^n(\sigma_1^n, \gamma_1^n)) = \frac{1}{n}\sum_{i=1}^{n} d(x_i, y_i, z(\sigma_i, \gamma_i)) = 0$$

for $x_1^n \in T^n_{[P_X]_\epsilon}$ and $y_1^n \in T^n_{[P_Y]_\epsilon}$. Next, suppose $x_1^n \notin T^n_{[P_X]_\epsilon}$ and $y_1^n \in T^n_{[P_Y]_\epsilon}$. Then, according to the encoding strategy outlined above, $x_1^n$ is directly conveyed to Merlin. On knowing $x_1^n$ he chooses *any* $\sigma_1^n$ such that $P_X(x_i)p(\sigma_i|x_i) > 0$ for all $i = 1, \ldots, n$, and reproduces $z^n(\sigma_1^n, \gamma_1^n)$. Again, (17) implies that $d(x_1^n, y_1^n, z^n(\sigma_1^n, \gamma_1^n)) = 0$. The other cases are handled similarly. Thus, condition (13) is satisfied.

*Converse:* Let $(\phi_X, \phi_Y, \psi)$ be any prefix-free code satisfying (13) for some $n \geq 1$. Let $(R_X, R_Y)$ be the corresponding rate pair. Define new random variables $\Sigma = \phi_X(X_1^n)$ and $\Gamma = \phi_Y(Y_1^n)$. Then

$$R_X \geq \frac{H(\Sigma)}{n} = \frac{I(X_1^n; \Sigma)}{n} \geq \frac{1}{n}\sum_{i=1}^{n} I(X_i; \Sigma)$$

and similarly for $R_Y$. Now

$$\begin{aligned}&\Pr[X_i = x, Y_i = y, \Sigma = \sigma, \Gamma = \gamma]\\ &\quad = P(x,y)\Pr[\Sigma = \sigma|X_i = x]\Pr[\Gamma = \gamma|Y_i = y].\end{aligned}$$

Since $P(x, y) > 0$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, and $(\phi_X, \phi_Y, \psi)$ satisfies (13), we have

$$d(x, y, \psi_i(\sigma, \gamma)) = 0$$

whenever $\Pr[\Sigma = \sigma|X_i = x]\Pr[\Gamma = \gamma|Y_i = y] > 0$. Defining the random variable $Q$ by $p(q) = \frac{1}{n}$ for $q \in \{1, 2, \ldots, n\}$, and setting

$$\begin{aligned}p(x|q) &= P_X(x)\\ p(\sigma|x,q) &= \Pr[\Sigma = \sigma|X_q = x]\\ p(y|q) &= P_Y(y)\\ p(\gamma|y,q) &= \Pr[\Gamma = \gamma|Y_q = y]\end{aligned}$$

we obtain

$$I(X; \Sigma|Q) = \frac{1}{n}\sum_{i=1}^{n} I(X_i; \Sigma) \leq R_X$$

$$I(Y; \Gamma|Q) = \frac{1}{n}\sum_{i=1}^{n} I(Y_i; \Gamma) \leq R_Y.$$

Also, by setting $z_q(\sigma, \gamma) = \psi_q(\sigma, \gamma)$, we have for any $(x, y, \sigma, \gamma, q)$

$$p(\sigma|x,q)p(\gamma|y,q)d(x, y, z_q(\sigma, \gamma)) = 0$$

and, therefore, (16), or equivalently (15), follows. □

### B. Coding for Remote Computing

Let $P(x, y)$ be an arbitrary distribution on $\mathcal{X} \times \mathcal{Y}$ with $P(x, y) > 0$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, and such that the corresponding marginals are $P_X$ and $P_Y$ respectively. Let the i.i.d. random variables $(X_1, Y_1), (X_2, Y_2), \ldots,$ be drawn from $P(x, y)$. Suppose that instead of the individual values of $(X_1, Y_1), (X_2, Y_2), \ldots,$ Merlin wishes to evaluate $f(X_1, Y_1), f(X_2, Y_2), \ldots,$ where $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is an arbitrary partial function. (Thus, we allow $f$ to be undefined for some arguments $(x, y)$.) We require that Merlin evaluate $f(x, y)$ correctly for every realization $(x, y) \in \mathcal{X} \times \mathcal{Y}$ where $f$ is defined, but we do not care about Merlin's reconstruction when $f$ is not defined.

The treatment in the previous section can be directly applied to this problem upon defining the distortion measure $d_f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to [0, 1]$ by

$$d_f(x,y,z) = \begin{cases} 1, & \text{if } f(x,y) \text{ defined, and } f(x,y) \neq z\\ 0, & \text{else}\end{cases}$$

and extending this definition to vectors $(x_1^n, y_1^n, z_1^n)$ as in (12). Therefore, the achievable rate region is given by

$$\mathcal{R}_f(P_X, P_Y) \stackrel{\text{def}}{=} \mathcal{R}_{d_f}(P_X, P_Y).$$

We will now provide a more intuitive reformulation of the characterization of $\mathcal{R}_f(P_X, P_Y)$, which brings out its dependence on $f$ explicitly.

*Definition 6:* Let $\mathcal{C}^f = (\mathcal{C}_X^f, \mathcal{C}_Y^f)$, where $\mathcal{C}_X^f$ and $\mathcal{C}_Y^f$ are collections of subsets of $\mathcal{X}$ and $\mathcal{Y}$, respectively. $\mathcal{C}^f$ is an *$f$-cover* of $\mathcal{X} \times \mathcal{Y}$ if

1) every $x \in \mathcal{X}$ ($y \in \mathcal{Y}$) is contained in some $s \in \mathcal{C}_X^f$ ($t \in \mathcal{C}_Y^f$) and
2) associated with every $s \in \mathcal{C}_X^f$ and $t \in \mathcal{C}_Y^f$ is a unique value $z_{st}$ such that $f(x, y) = z_{st}$ for all pairs $(x, y) \in s \times t$ for which $f(x, y)$ is defined.

*Theorem 3:* Let $\mathcal{R}'$ be the closure of the set of rate pairs $(R_X, R_Y)$, where

$$R_X > I(X; S|Q) \tag{19}$$

$$R_Y > I(Y; T|Q) \tag{20}$$

for some choice of the joint distribution

$$p(q)p(s|x,q)p(t|y,q)P_X(x)P_Y(y)$$

which satisfies the following conditions. For each value $Q = q$, let $\mathcal{C}_q^f = (\mathcal{C}_{X,q}^f, \mathcal{C}_{Y,q}^f)$ be an $f$-cover of $\mathcal{X} \times \mathcal{Y}$. For every $x \in \mathcal{X}$, $p(s|x, q) > 0$ only if $x \in s \in \mathcal{C}_{X,q}^f$. $p(t|y, q)$ is chosen similarly. (We may assume that $Q$ is distributed over $\{1, 2, 3\}$.) Then $\mathcal{R}' = \mathcal{R}_f(P_X, P_Y)$.

*Proof:* For each $q$, Let $\mathcal{C}_q^f = (\mathcal{C}_{X,q}^f, \mathcal{C}_{Y,q}^f)$ be an $f$-cover, and let $(p(s|x, q), p(t|y, q))$ be a pair of distributions such that

$$p(s|x,q) > 0 \Longrightarrow x \in s \in \mathcal{C}_{X,q}^f$$

and

$$p(t|y,q) > 0 \Longrightarrow y \in t \in \mathcal{C}_{Y,q}^f.$$

Then $(I(X; S|Q), I(Y; T|Q)) \in \mathcal{R}'$. For any pair $(s, t) \in \mathcal{C}_{X,q}^f \times \mathcal{C}_{Y,q}^f$, set

$$z_q(s,t) = \begin{cases} f(x,y), & \text{if } f(x,y) \text{ is defined}\\ & \quad \text{for } some\ (x,y) \in s \times t\\ \text{undefined}, & \text{else.}\end{cases}$$

Since, by definition of an $f$-cover, $f(x, y)$ takes a unique value for every $(x, y) \in s \times t$ for which it is defined, $z_q$ is a well-defined (partial) function. Further, for any $(x, y, s, t, q)$

$$\begin{aligned}p(s|x,q)p(t|y,q) > 0 &\Longrightarrow (x,y) \in s \times t\\ &\Longrightarrow d_f(x, y, z_q(s,t)) = 0\end{aligned}$$

by the definition of $d_f$. Thus, $p(s|x, q)$, $p(t|y, q)$, and $z_q(s, t)$ playing the roles of $p(\sigma|x, q)$, $p(\gamma|y, q)$, and $z_q(\sigma, \gamma)$, respectively, (16) is satisfied. Thus,

$$(I(X; S|Q), I(Y; T|Q)) \in \mathcal{R}_{d_f}(P_X, P_Y) = \mathcal{R}_f(P_X, P_Y).$$

Conversely, let $p(\sigma|x,q)$, $p(\gamma|y,q)$, and $z_q(\sigma,\gamma)$ satisfy (16), so that $(I(X;\Sigma|Q), I(Y;\Gamma|Q)) \in \mathcal{R}_f(P_X,P_Y)$. Also, let

$$\mathcal{X}_q(\sigma) \stackrel{\text{def}}{=} \{x : p(\sigma|x,q) > 0\}$$
$$\mathcal{Y}_q(\gamma) \stackrel{\text{def}}{=} \{y : p(\gamma|y,q) > 0\}.$$

Then

$$d_f(x,y,z_q(\sigma,\gamma)) = 0, \qquad \text{for all } (x,y) \in \mathcal{X}_q(\sigma) \times \mathcal{Y}_q(\gamma). \tag{21}$$

Observe that for all pairs $(x,y) \in \mathcal{X}_q(\sigma) \times \mathcal{Y}_q(\gamma)$, $f(x,y)$ either assumes a constant value whenever defined, or is not defined at all. (Otherwise, it is impossible to satisfy (21) with a unique $z_q(\sigma,\gamma)$.) Thus, if $\mathcal{X}_q(\sigma) = \mathcal{X}_q(\sigma')$ then for all $\gamma$, either $z_q(\sigma,\gamma) = z_q(\sigma',\gamma)$, or both $z_q(\sigma,\gamma)$ and $z_q(\sigma',\gamma)$ can be left undefined. Similarly when $\mathcal{Y}_q(\gamma) = \mathcal{Y}_q(\gamma')$.

Now merge all $\sigma$ with identical $\mathcal{X}_q(\sigma)$ and all $\gamma$ with identical $\mathcal{Y}_q(\gamma)$. Set $p(s|x,q) = p(\sigma|x,q)$, where $s = \mathcal{X}_q(\sigma)$, and let $\mathcal{C}^f_{X,q}$ be the collection of sets $\mathcal{X}_q(\sigma)$. Similarly, define sets $t$, distributions $p(t|y,q)$, and the collection $\mathcal{C}^f_{Y,q}$. We have then proved that $(\mathcal{C}^f_{X,q}, \mathcal{C}^f_{Y,q})$ is an $f$-cover, and

$$(I(X;\Sigma|Q), I(Y;\Gamma|Q)) \in \mathcal{R}'. \qquad \square$$

### C. A Single-Letter Inner Bound for $\mathcal{R}^{\text{vl}}(G,P_X,P_Y)$

Define the partial function $e : \mathcal{X} \times \mathcal{Y} \longrightarrow E$ by

$$e(x,y) = \begin{cases} \{x,y\}, & \text{if } \{x,y\} \in E \\ \text{undefined}, & \text{else.} \end{cases} \tag{22}$$

Then $\{x,y\}$ and $\{x',y'\}$ are distinct edges in $G$ if and only if $e(x,y)$ and $e(x',y')$ are both defined, and $e(x,y) \neq e(x',y')$. Thus, for $n=1$, there exists a one-to-one correspondence between zero-error prefix-free codes for $G$ and prefix-free codes which enable Merlin to compute the function $e$ (in the sense of (13) with $d = d_e$.) For $n > 1$, on the other hand, the triplet $(\phi_X,\phi_Y,\psi)$ constitutes a valid zero-error code for $G$ only if, for distinct $(x_1^n, y_1^n)$ and $(x_1'^n, y_1'^n)$

$$\begin{aligned} &\{x_i,y_i\} \in E \text{ and } \{x_i',y_i'\} \in E \text{ for } all \ 1 \le i \le n \\ &\quad \Longrightarrow \psi(\phi_X(x_1^n), \phi_Y(y_1^n)) \neq \psi(\phi_X(x_1'^n), \phi_Y(y_1'^n)). \end{aligned} \tag{23}$$

However, for a valid code $(\phi_X,\phi_Y,\psi)$ which evaluates the function $e$, if $e(x_i,y_i)$ and $e(x_i',y_i')$ are both defined and $e(x_i,y_i) \neq e(x_i',y_i')$ for some $1 \le i \le n$, then

$$\psi(\phi_X(x_1^n), \phi_Y(y_1^n)) \neq \psi(\phi_X(x_1'^n), \phi_Y(y_1'^n)).$$

In other words, $(\phi_X,\phi_Y,\psi)$ satisfies the necessary condition

$$\begin{aligned} &\{x_i,y_i\} \in E, \{x_i',y_i'\} \in E, \text{ and } (x_i,y_i) \neq (x_i',y_i') \\ &\qquad \text{for } some \ 1 \le i \le n \\ &\quad \Longrightarrow \psi(\phi_X(x_1^n), \phi_Y(y_1^n)) \neq \psi(\phi_X(x_1'^n), \phi_Y(y_1'^n)). \end{aligned} \tag{24}$$

But (24) is a stricter requirement than (23). Thus, we have that every code $(\phi_X,\phi_Y,\psi)$ satisfying (13) with $d = d_e$ is also a zero-error code for $G$. Since the rate region for the codes satisfying (13) depends only on the marginals $P_X$ and $P_Y$, we may assume for computation purposes that $P_{XY}(x,y) = P_X(x)P_Y(y)$, thus guaranteeing $P_{XY}(x,y) > 0$ for all $(x,y) \in \mathcal{X} \times \mathcal{Y}$. Theorem 3 then provides a characterization for the region $\mathcal{R}_e(P_X,P_Y)$. Note that, for the function $e$, $e$-covers of $\mathcal{X} \times \mathcal{Y}$ are the same as bipartite covers of $G$.

Thus, we have proved that

$$\mathcal{R}^{\text{in}}(G,P_X,P_Y) \stackrel{\text{def}}{=} \mathcal{R}_e(P_X,P_Y) \subseteq \mathcal{R}^{\text{vl}}(G,P_X,P_Y). \tag{25}$$

### D. A Single-Letter Outer Bound for $\mathcal{R}^{\text{vl}}(G,P_X,P_Y)$

The result of Slepian and Wolf [22] will be relevant to us here, so we begin by briefly summarizing it. Let $(A,B)$ be a pair of finite random variables distributed over $\mathcal{A} \times \mathcal{B}$ according to $P(a,b)$. Fix $\epsilon > 0$. There exists a code $(f_A, f_B, g)$, with $f_A : \mathcal{A}^n \to \{0,1\}^*$, $f_B : \mathcal{B}^n \to \{0,1\}^*$, and $g : \{0,1\}^* \times \{0,1\}^* \to \mathcal{A}^n \times \mathcal{B}^n$, of rates

$$R_A = \frac{1}{n} \sum P^n(a_1^n, b_1^n) |f_A(a_1^n)|$$
$$R_B = \frac{1}{n} \sum P^n(a_1^n, b_1^n) |f_B(b_1^n)|$$

such that

$$\Pr\left[\{(a_1^n,b_1^n) \in \mathcal{A}^n \times \mathcal{B}^n : g(f_A(a_1^n), f_B(b_1^n)) = (a_1^n, b_1^n)\}\right] > 1 - \epsilon \tag{26}$$

if and only if

$$R_A \ge H(A|B)$$
$$R_B \ge H(B|A)$$
$$R_A + R_B \ge H(A,B).$$

Consider any prefix-free zero-error code $(\phi_X,\phi_Y,\psi)$ for $G$. Such a code satisfies (26) for $(A,B) = (X',Y')$, with

$$(f_A, f_B, g) = (\phi_X, \phi_Y, \psi)$$

for any distribution $P(x,y)$ such that $\{X',Y'\} \in E$, $X' \sim P_X$, and $Y' \sim P_Y$. Thus, if $(R_X,R_Y) \in \mathcal{R}^{\text{vl}}(G,P_X,P_Y)$, then

$$R_X + R_Y \ge \max_{\{X',Y'\} \in E, X' \sim P_X, Y' \sim P_Y} H(X',Y'). \tag{27}$$

Next, note that

$$\phi_X(x_1^n) \neq \phi_X(x_1'^n)$$

for every distinct $x_1^n, x_1'^n \in \mathcal{X}^n$ such that $\{x_1^n, x_1'^n\} \in E_X^n$. More generally

$$\phi_X(x_1^n) \neq \phi_X(x_1'^n)$$

if $x_1^n, x_1'^n \in s_X^n$ for some $s_X^n \in \mathcal{T}(G_X)^n$, so that any $x_1^n \in \mathcal{X}^n$ can be recovered without error from knowledge of the pair $(\phi_X(x_1^n), s_X^n)$ with $x_1^n \in s_X^n \in \mathcal{T}(G_X)^n$. Thus, if $(A,B)$ is any random pair $(X, S_X)$ such that $X \in S_X \in \mathcal{T}(G_X)$, then $f_A = \phi_X$, together with $f_B$ conveying $S_X$ directly in a point-to-point lossless encoding sense, constitute a valid encoder pair satisfying (26). This shows that if $(R_X,R_Y) \in \mathcal{R}^{\text{vl}}(G,P_X,P_Y)$, then

$$R_X \ge \max_{X \in S_X \in \mathcal{T}(G_X)} H(X|S_X). \tag{28}$$

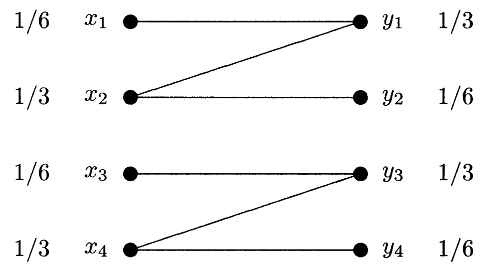



Fig. 8. $(G, P_X, P_Y)$ for the example of Section III-F. Marginal probabilities are marked next to the corresponding node labels.

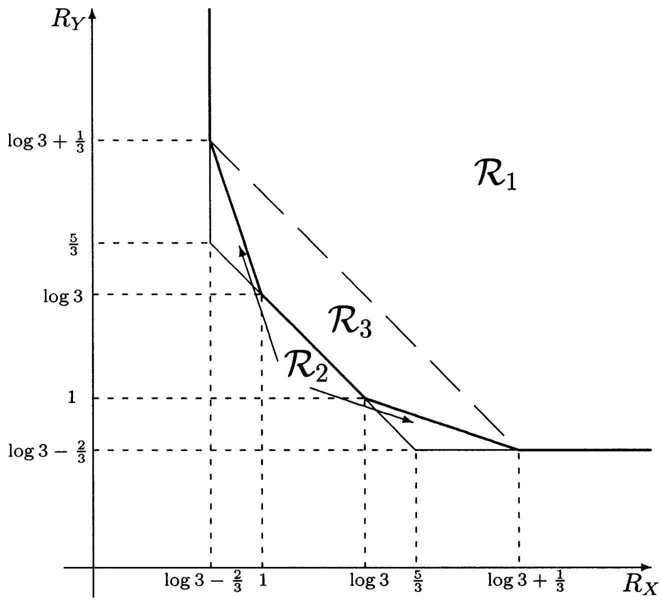



Fig. 9. $\mathcal{R}^{\rm out}(G, P_X, P_Y)$ and $\mathcal{R}^{\rm in}(G, P_X, P_Y)$ for $(G, P_X, P_Y)$ in Fig. 8. $\mathcal{R}^{\rm out}(G, P_X, P_Y) = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, and $\mathcal{R}_1 \cup \mathcal{R}_3 \subseteq \mathcal{R}^{\rm in}(G, P_X, P_Y)$.

Similarly

$$R_Y \geq \max_{Y \in T_Y \in \mathcal{T}(G_Y)} H(Y|T_Y). \tag{29}$$

We have thus derived an outer bound

$$\mathcal{R}^{\rm out}(G, P_X, P_Y) \supseteq \mathcal{R}^{\rm vl}(G, P_X, P_Y)$$

in (27)–(29).

### *E. Computation of the Bounds for an Example*

Let us calculate $\mathcal{R}^{\rm out}(G, P_X, P_Y)$ and $\mathcal{R}^{\rm in}(G, P_X, P_Y)$ for a particular example, shown in Fig. 8. Consider first $\mathcal{R}^{\rm out}(G, P_X, P_Y)$. Clearly

$$\max_{\{X', Y'\} \in E, X' \sim P_X, Y' \sim P_Y} H(X', Y') = \log 6$$

with the maximum being achieved by the distribution which assigns probability $1/6$ to each edge. We also obtain

$$\max_{X \in S_X \in \mathcal{T}(G_X)} H(X|S_X) = \log 3 - 2/3$$

by setting $P(s_X|x) = 1$ for

$$x \in s_X \in \mathcal{T}(G_X) = \{\{x_1, x_2\}, \{x_3, x_4\}\}.$$

By symmetry

$$\max_{Y \in T_Y \in \mathcal{T}(G_Y)} H(Y|T_Y) = \log 3 - 2/3$$

as well. Thus,

$$\mathcal{R}^{\rm out}(G, P_X, P_Y) = \left\{ (R_X, R_Y) : \ R_X \geq \log 3 - \frac{2}{3}, \right. \\ \left. R_Y \geq \log 3 - \frac{2}{3}, R_X + R_Y \geq \log 6 \right\} \tag{30}$$

i.e., it is the region $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ in Fig. 9.

Now for $\mathcal{R}^{\rm in}(G, P_X, P_Y)$. Note that $\mathcal{C} = (\mathcal{C}_X, \mathcal{C}_Y)$, where

$$\mathcal{C}_X = \{\{x_1, x_3\}, \{x_2\}, \{x_4\}\}$$

and

$$\mathcal{C}_Y = \{\{y_1, y_4\}, \{y_2, y_3\}\}$$

is a bipartite cover of $G$. Choose, for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, the distributions $p(s|x) = 1$ if $x \in s$, and $p(t|y) = 1$ if $y \in t$, over $s \in \mathcal{C}_X$ and $t \in \mathcal{C}_Y$, respectively. (This makes sense, since each $x$ $(y)$ belongs to a unique $s \in \mathcal{C}_X$ $(t \in \mathcal{C}_Y)$.) This shows that

$$(I(X;S), I(Y;T)) = (\log 3, 1) \in \mathcal{R}^{\rm in}(G, P_X, P_Y).$$

By symmetry, $(1, \log 3) \in \mathcal{R}^{\rm in}(G, P_X, P_Y)$ as well.

Consider, next, the "corner" points of $\mathcal{R}^{\rm in}(G, P_X, P_Y)$. Alice directly encodes $X$, incurring the rate $H(X) = 1/3 + \log 3$. Since Merlin can decode $X$ without any error, Bob only needs to distinguish between $y_1$ and $y_2$, and between $y_3$ and $y_4$. Bob can do this by assigning the same codeword to $y_1$ and $y_3$, and $y_2$ and $y_4$, and this entails a rate of

$$-(1/6 + 1/6)\log(1/6 + 1/6) \\ -(1/3 + 1/3)\log(1/3 + 1/3) = \log 3 - 2/3.$$

Thus, $(\log 3 + 1/3, \log 3 - 2/3) \in \mathcal{R}^{\rm in}(G, P_X, P_Y)$. The achievability of this point can also be seen by choosing

$$\mathcal{C} = (\{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}\}, \{\{y_1, y_3\}, \{y_2, y_4\}\})$$

and setting $p(s|x) = 1$ if $x \in s$, and $p(t|y) = 1$ if $y \in t$.

By time-sharing the above points, we see that $(R_X, R_Y) \in \mathcal{R}^{\rm in}(G, P_X, P_Y)$ if it can be expressed in the form

$$\alpha(1, \log 3) + (1 - \alpha)(\log 3, 1) \\ \text{or } \alpha(\log 3, 1) + (1 - \alpha)(\log 3 + 1/3, \log 3 - 2/3) \\ \text{or } \alpha(\log 3 - 2/3, \log 3 + 1/3) + (1 - \alpha)(1, \log 3)$$

for some $0 \leq \alpha \leq 1$. Thus, $\mathcal{R}_1 \cup \mathcal{R}_3 \subseteq \mathcal{R}^{\rm in}(G, P_X, P_Y)$ in Fig. 9.

Let us point out a couple of noteworthy features of the bounds calculated above.

1) $\mathcal{R}^{\rm in}(G, P_X, P_Y)$ and $\mathcal{R}^{\rm out}(G, P_X, P_Y)$ *coincide* in the range $1 \leq R_X \leq \log 3, 1 \leq R_Y \leq \log 3$. Thus, the bounds yield a tight characterization of $\mathcal{R}^{\rm vl}(G, P_X, P_Y)$ in this range. Also, the corner points

$$(\log 3 - 2/3, \log 3 + 1/3) \ \text{ and } \ (\log 3 + 1/3, \log 3 - 2/3)$$

are on the boundary of $\mathcal{R}^{\rm vl}(G,P_X,P_Y)$. The characterization of $\mathcal{R}^{\rm vl}(G,P_X,P_Y)$ in the remaining ranges

$$\log 3-2/3<R_X<1 \quad\text{and}\quad \log 3<R_X<\log 3+1/3$$

continues to be unknown.

2) The rate region achievable by time-sharing the corner points

$$(\log 3-2/3, \log 3+1/3) \quad\text{and}\quad (\log 3+1/3, \log 3-2/3)$$

i.e., the region $\mathcal{R}_1$, is *strictly* contained in $\mathcal{R}^{\rm in}(G,P_X,P_Y)$. Thus, successive encoding combined with time sharing is not an optimal encoding strategy for this example (except for the trivial cases corresponding to no time sharing.) This may be contrasted with the results of Slepian and Wolf [22], which show that successive encoding combined with time sharing yields all points on the achievable rate region for any correlated source when the zero-error constraint is relaxed to requiring a vanishingly small probability of error.

### F. Implications on the Unknown Side Information Problem

Define

$$R_X^{\rm vl}(G,P_X,P_Y) \stackrel{\rm def}{=} \min_{(R_X,R_Y)\in\mathcal{R}^{\rm vl}(G,P_X,P_Y)} R_X \tag{31}$$

which indicates the minimum rate achieved by successive encoding. It follows from Corollary 1, (6), and (7) that

$$\begin{aligned} R_X^{\rm vl}(G,P_X,P_Y) &= L(G,P_X,P_Y,1) \\ &= \lim_{n\to\infty}\frac{1}{n}\min_{\Phi_X,\Phi_Y} H(\Phi_X(X^n)) \end{aligned}$$

where the minimization above is over all bipartite colorings of $G^n$. Here, a key observation is that if $(\Phi_X,\Phi_Y)$ is a bipartite coloring of $G^n$, then $\Phi_X$ is also a coloring of $G_X^n$, as no pair of $\mathcal{X}$-nodes in a color class $\Phi_X^{-1}(i)$ can be connected to the same $\mathcal{Y}$-node in $G^n$, and hence to each other in $G_X^n$. Conversely, if $\Phi_X$ is a coloring of $G_X^n$ then $(\Phi_X,\Upsilon_Y)$, where the identity coloring $\Upsilon_Y$ assigns a different color to each $y_1^n$, is a bipartite coloring of $G^n$. Therefore,

$$\begin{aligned} R_X^{\rm vl}(G,P_X,P_Y) &= R_X^{\rm vl}(G_X,P_X) \\ &= \lim_{n\to\infty}\frac{1}{n}\min_{\Phi_X} H(\Phi_X(X^n)) \end{aligned} \tag{32}$$

where the minimization is over all colorings of $G_X^n$. Note that we emphasize in (32) the fact that the dependence of $R_X^{\rm vl}(G,P_X,P_Y)$ on $G$ is fully captured by $G_X$, and further, it does not depend on $P_Y$ at all. In [1], the minimum of (32) was defined as the chromatic entropy of $G_X^n$, denoted $H_\chi(G_X^n,P_X^n)$, and (32) was proven for prefix-free codes. The result was later generalized to the whole variable-length coding class in [17, Lemma 2].

Let us also define

$$R_X^{\rm in}(G,P_X,P_Y) \stackrel{\rm def}{=} \min_{(R_X,R_Y)\in\mathcal{R}^{\rm in}(G,P_X,P_Y)} R_X \tag{33}$$

$$R_X^{\rm out}(G,P_X,P_Y) \stackrel{\rm def}{=} \min_{(R_X,R_Y)\in\mathcal{R}^{\rm out}(G,P_X,P_Y)} R_X. \tag{34}$$

It immediately follows that

$$R_X^{\rm in}(G,P_X,P_Y) \ge R_X^{\rm vl}(G,P_X,P_Y) \ge R_X^{\rm out}(G,P_X,P_Y). \tag{35}$$

A similar argument as the above for bipartite colorings can be repeated for bipartite covers. Specifically, if $(\mathcal{C}_X,\mathcal{C}_Y)$ is a bipartite cover of $G$, then $\mathcal{C}_X$ is also a covering of $G_X$ with sets from $\mathcal{T}(\overline{G_X})$. Note that $\mathcal{T}(\overline{G_X})$ is a collection of maximal subgraphs of $G_X$ which do not contain any edges. Conversely, any covering of $G_X$ with members of $\mathcal{T}(\overline{G_X})$ can be combined with the identity covering $\mathcal{I}_Y$ to form a bipartite cover of $G$. Therefore, the discussion in Section III-D implies

$$\begin{aligned} R_X^{\rm in}(G,P_X,P_Y) &= R_X^{\rm in}(G_X,P_X) \\ &= \min_{X\in S_X\in\mathcal{T}(\overline{G_X})} I(X;S_X). \end{aligned} \tag{36}$$

The right-hand side of (36) was defined by Körner [12] as the *graph entropy*, denoted $H(G_X,P_X)$, and was already shown to be an upper bound to $R_X^{\rm vl}(G_X,P_X)$ in [1]. It was also shown in [1] that graph entropy characterizes the *exact* minimum achievable rate in a constrained variable-length coding scheme for the unknown side information problem.

Finally, the results of Section III-E imply that

$$\begin{aligned} R_X^{\rm out}(G,P_X,P_Y) &= R_X^{\rm out}(G_X,P_X) \\ &= \max_{X\in S_X\in\mathcal{T}(G_X)} H(X|S_X) \end{aligned} \tag{37}$$

which was obtained as a lower bound to $R_X^{\rm vl}(G_X,P_X)$ in [17, Lemma 3]. The maximization in (37) was defined in [1] as the *clique entropy* of the graph $G_X$. Since $H(X|S_X)=H(X)-I(X;S_X)$, (37) can alternatively be written as

$$R_X^{\rm out}(G_X,P_X)=H(X)-H(\overline{G_X},P_X). \tag{38}$$

From (35), (36), and (38), it follows that

$$H(G_X,P_X)\ge H(X)-H(\overline{G_X},P_X). \tag{39}$$

Now, let us fix the graph $G_X$, and allow the distribution $P_X$ to vary. In [14], Körner and Longo initiated the study of the following questions. They were motivated by a different, apparently unrelated, two-step source-coding problem, but the relevance of the questions to the successive encoding problem is clear.

1) What condition on $G_X$ guarantees that there exists a nonvanishing distribution $P_X$ on the vertex set of $G_X$ (i.e., $P_X(x)>0$ for all $x\in\mathcal{X}$) such that equality is achieved in (39)?

2) What condition on $G_X$ guarantees equality in (39) for *all* distributions on $\mathcal{X}$?

Subsequent investigations revealed a surprisingly deep interplay between purely combinatorial properties of $G_X$ and the information-theoretic question of equality in (39). The answer to Question 1 is affirmative if and only if $G_X$ is a *normal* graph [15]. On the other hand, Question 2 has an affirmative answer if and only if $G_X$ is *perfect* [9]. The graph $G_X$ is normal if its vertex set can be covered by a collection of sets from $\mathcal{T}(G_X)$ as well as a collection from $\mathcal{T}(\overline{G_X})$, such that any pair of sets, one from each collection, share a common vertex [14]. $G_X$ is said to be perfect if for every subgraph $G'_X$ of $G_X$, $\chi(G'_X) = \omega(G'_X)$ is satisfied [3, Ch. 16]. Having its origins in information theory, the important class of perfect graphs attracted many researchers mainly due to Berge's long-standing *strong perfect graph conjecture* [4]. It follows from the answers to Questions 1 and 2 that all perfect graphs are normal. This fact was also directly proven in [13].

In the next section, in search for answers to information-theoretic questions similar to the above regarding bipartite graphs and zero-error correlated source coding, we will derive purely combinatorial conditions on $G$, thus extending the results of [9], [15].

Another interesting result regarding the bounds for the unknown side information problem was shown in [1]: $R_X^{\text{in}}(G_X, P_X)$ can be significantly larger than $R_X^{\text{out}}(G_X, P_X)$. Namely, there are graphs $G_X$ with arbitrarily large number of nodes, such that

$$R_X^{\text{out}}(G_X, P_X) \le \frac{1}{2} R_X^{\text{in}}(G_X, P_X) + o(R_X^{\text{in}}(G_X, P_X)).$$

This result implies that there are bipartite graphs $G$ where the gap between $\mathcal{R}^{\text{out}}(G, P_X, P_Y)$ and $\mathcal{R}^{\text{in}}(G, P_X, P_Y)$ is arbitrarily large.

## IV. Tightness of the Asymptotic Bounds

In this section, we derive conditions for the coincidence of $\mathcal{R}^{\text{out}}(G, P_X, P_Y)$ and $\mathcal{R}^{\text{in}}(G, P_X, P_Y)$ based on purely combinatorial properties of $G$.

Let $R_Y^{\text{in}}(G_Y, P_Y)$ and $R_Y^{\text{out}}(G_Y, P_Y)$ be defined similarly to $R_X^{\text{in}}(G_X, P_X)$ and $R_X^{\text{out}}(G_X, P_X)$ of (33) and (34), respectively. Also define

$$R_T^{\text{in}}(G, P_X, P_Y) \stackrel{\text{def}}{=} \min_{(R_X, R_Y) \in \mathcal{R}^{\text{in}}(G, P_X, P_Y)} R_X + R_Y \tag{40}$$

and

$$R_T^{\text{out}}(G, P_X, P_Y) \stackrel{\text{def}}{=} \min_{(R_X, R_Y) \in \mathcal{R}^{\text{out}}(G, P_X, P_Y)} R_X + R_Y \tag{41}$$

as the minimum *total* rates in $\mathcal{R}^{\text{in}}(G, P_X, P_Y)$ and $\mathcal{R}^{\text{out}}(G, P_X, P_Y)$, respectively. From (19), (20), (27), (28), and (29), we obtain

$$R_T^{\text{in}}(G, P_X, P_Y) = \min_{\mathcal{C}} \min_{X \in S \in \mathcal{C}_X, Y \in T \in \mathcal{C}_Y} I(X;S) + I(Y;T) \tag{42}$$

$$R_T^{\text{out}}(G, P_X, P_Y) = \max\left\{ \max_{\{X', Y'\} \in E, X' \sim P_X, Y' \sim P_Y} H(X', Y'), \; R_X^{\text{out}}(G_X, P_X) + R_Y^{\text{out}}(G_Y, P_Y) \right\}. \tag{43}$$

where the outer minimization in (42) is over the bipartite covers of $G$. Note that both the maximum of (43) and the minimum of (42) exist, since the respective constraint sets

$$\left\{ P(x,y) : P(x,y) = 0 \text{ if } \{x,y\} \notin E, \; \sum_{y'} P(x, y') = P_X(x), \sum_{x'} P(x', y) = P_Y(y) \right\}$$

and

$$\left\{ (p(s|x), p(t|y)) : X \in S \in \mathcal{C}_X, Y \in T \in \mathcal{C}_Y \right\}$$

are compact, and the functions $H(X', Y')$ and $I(X;S) + I(Y;T)$ are continuous in their respective arguments $P(x,y)$ and $(p(s|x), p(t|y))$.

The fact that $\mathcal{R}^{\text{out}}(G, P_X, P_Y) \supseteq \mathcal{R}^{\text{in}}(G, P_X, P_Y)$ trivially implies that $R_T^{\text{in}}(G, P_X, P_Y) \ge R_T^{\text{out}}(G, P_X, P_Y)$, and hence that

$$\min_{\mathcal{C}} \min_{X \in S \in \mathcal{C}_X, Y \in T \in \mathcal{C}_Y} I(X;S) + I(Y;T) \ge \max_{\{X', Y'\} \in E, X' \sim P_X, Y' \sim P_Y} H(X', Y'). \tag{44}$$

However, if we ignore the "coding interpretation," then the validity of inequality (44) is not obvious. In the following lemma, we give a direct proof of (44) without recourse to the coding interpretation $R_T^{\text{in}}(G, P_X, P_Y) \ge R_T^{\text{out}}(G, P_X, P_Y)$. The lemma also yields conditions for equality in (44), which we will later use to demonstrate the fact that the combinatorial structure of $G$ alone provides extensive information about the coincidence of $\mathcal{R}^{\text{in}}(G, P_X, P_Y)$ and $\mathcal{R}^{\text{out}}(G, P_X, P_Y)$.

*Lemma 2:* Let $G$, $P_X$, and $P_Y$ be given. Let $\mathcal{C} = (\mathcal{C}_X, \mathcal{C}_Y)$ be a bipartite cover of $G$, and random variables $X, Y, S$, and $T$ be jointly distributed according to

$$P_{XYST}(x, y, s, t) = p(s|x) p(t|y) P(x,y) \tag{45}$$

where $X \in S \in \mathcal{C}_X$, $Y \in T \in \mathcal{C}_Y$, $\{X, Y\} \in E$, $X \sim P_X$, and $Y \sim P_Y$. Then

$$I(X;S) + I(Y;T) \ge H(X,Y). \tag{46}$$

Equality holds if and only if the distributions are such that

$$\sum_{x \in s, y \in t} p(s|x) p(t|y) P(x,y) = \left\{ \sum_{x \in s} p(s|x) P_X(x) \right\} \left\{ \sum_{y \in t} p(t|y) P_Y(y) \right\} \tag{47}$$

for all $s \in \mathcal{C}_X, t \in \mathcal{C}_Y$. Thus, equality holds in (44) if and only if there exists a bipartite cover $\mathcal{C}$ and a joint distribution $P_{XYST}$ as in (45) satisfying (47).

*Proof:* For a fixed $(s,t)$, $P(x,y) > 0$ for at most one pair $(x,y) \in s \times t$, since any pair $(s,t)$ induces at most one edge in $G$. Thus, whenever $P_{ST}(s,t) > 0$, $P_{XY|ST}(x, y|s, t)$ takes only the values 0 and 1, and $H(X, Y|S, T) = 0$. Further

$$P_{ST|XY}(s, t|x, y) = p(s|x) p(t|y)$$

which implies

$$H(S, T|X, Y) = H(S|X) + H(T|Y).$$

Using these relations and the trivial inequality $H(S) + H(T) \ge H(S,T)$, we have

$$\begin{aligned} I(X;S) + I(Y;T) &= H(S) - H(S|X) + H(T) - H(T|Y) \\ &\ge H(S,T) - H(S,T|X,Y) \\ &= H(X,Y) - H(X,Y|S,T) \\ &= H(X,Y) \end{aligned}$$

and (46) is proved. Equality is achieved if and only if $H(S) + H(T) = H(S,T)$, and the obvious necessary and sufficient condition is statistical independence

$$P_{ST}(s,t) = P_S(s)P_T(t) \text{ for all } (s,t) \in \mathcal{C}$$

which is, in fact, the condition specified in (47). □

Theorem 4 below investigates a question analogous to Question 1 of Section III-F. Namely, it states necessary and sufficient conditions on whether or not there exist nonvanishing marginals (i.e., $P_X(x) > 0$, $P_Y(y) > 0$ for all $(x,y)$) on $G$ such that (44) is satisfied with equality. Building on the result of Theorem 4, Theorem 5 provides sufficient conditions for the existence of nonvanishing marginals $P_X$ and $P_Y$ such that

$$\mathcal{R}^{\text{out}}(G,P_X,P_Y) = \mathcal{R}^{\text{in}}(G,P_X,P_Y).$$

Finally, Theorem 6 specifies conditions on $G$ to ensure equality in (44) for all marginals $P_X$ and $P_Y$, thus answering a question analogous to Question 2 of Section III-F. Throughout, we hold the bipartite graph $G$ fixed, and consider the class of all marginals $P_X$ and $P_Y$ on $G$. Recall that these are obtained by marginalization of joint distributions $P(x,y)$ on $\mathcal{X} \times \mathcal{Y}$ such that $P(x,y) > 0$ only if $\{x,y\} \in E$.

*Theorem 4:* Let $G = (\mathcal{X} \cup \mathcal{Y}, E)$ be an arbitrary bipartite graph. There exists some joint distribution $P(x,y)$ with nonvanishing marginals such that $G$ is the characteristic graph of $P$, and

$$\min_{\mathcal{C}} \min_{X\in S\in\mathcal{C}_X, Y\in T\in\mathcal{C}_Y} I(X;S) + I(Y;T) = \max_{\{X',Y'\}\in E, X'\sim P_X, Y'\sim P_Y} H(X',Y') \quad (48)$$

if and only if $G$ has an exact bipartite cover.

*Proof:* Suppose that (48) holds. If $p(s|x)$ and $p(t|y)$, such that $X \in S \in \mathcal{C}_X$ and $Y \in T \in \mathcal{C}_Y$ for the bipartite cover $\mathcal{C} = (\mathcal{C}_X, \mathcal{C}_Y)$, and $P(x,y)$, are the distributions which achieve the minimum and the maximum of (48), respectively, then (47) holds by Lemma 2. We claim that $\mathcal{C}$ is an exact bipartite cover of $G$. To show this by contradiction, suppose $\mathcal{C}$ is not exact. Then there exists some $s_0 \in \mathcal{C}_X$ and $t_0 \in \mathcal{C}_Y$ such that $s_0 \cup t_0$ induces no edges in $G$. For this $(s_0,t_0)$, the expression on the left-hand side in (47) vanishes, since $P(x,y) > 0$ only if $\{x,y\} \in E$. But the expression on the right-hand side does not vanish, since $p(s_0|x) > 0$ and $p(t_0|y) > 0$ for some $x \in s_0$ and $y \in t_0$, respectively, and $P_X(x) > 0$, $P_Y(y) > 0$ for all $(x,y)$.

Conversely, suppose $\mathcal{C}$ is an exact bipartite cover of $G$. Let $P_S(s)$ and $P_T(t)$ be any nonvanishing distributions on $s \in \mathcal{C}_X$ and $t \in \mathcal{C}_Y$, respectively. Define the joint distribution

$$P_{XYST}(x,y,s,t) = p(x,y|s,t)P_S(s)P_T(t) \quad (49)$$

on $\mathcal{X} \times \mathcal{Y} \times \mathcal{C}_X \times \mathcal{C}_Y$ by setting

$$p(x,y|s,t) = \begin{cases} 1, & \text{if } \{x,y\} \in E \text{ and } (x,y) \in s \times t \\ 0, & \text{else.} \end{cases} \quad (50)$$

Note that $p(x,y|s,t)$ is a valid distribution, since every $(s,t)$ induces exactly one edge. Let $P(x,y)$ and $P_{ST}(s,t)$ denote the respective marginals on $\mathcal{X} \times \mathcal{Y}$ and $\mathcal{C}_X \times \mathcal{C}_Y$, and $p(s|x)$, $p(s|x,y)$, $p(t|y)$, and $p(t|x,y)$ denote conditionals derived from $P_{XYST}(x,y,s,t)$. The claim follows from Lemma 2 once we verify that, for all $(x,y,s,t)$

$$P_{XYST}(x,y,s,t) = P(x,y)p(s|x)p(t|y) \quad (51)$$

$$P(x,y) > 0 \text{ only if } \{x,y\} \in E \quad (52)$$

$$p(s|x), p(t|y) > 0 \text{ only if } x \in s,\ y \in t \quad (53)$$

$$P_{ST}(s,t) = P_S(s)P_T(t). \quad (54)$$

Equations (52) and (54) follow easily from (50) and (49), respectively. Now, using (54) and (50), we get

$$P_{XYST}(x,y,s,t) = \begin{cases} P_S(s)P_T(t), & \text{if } \{x,y\} \in E \text{ and } (x,y) \in s \times t \\ 0, & \text{else.} \end{cases}$$

and

$$P(x,y) = \begin{cases} \sum_{s\ni x} P_S(s) \sum_{t\ni y} P_T(t), & \text{if } \{x,y\} \in E \\ 0, & \text{else.} \end{cases} \quad (55)$$

where the summations are over $s$ containing $x$, and $t$ containing $y$, respectively. So for $\{x,y\} \in E$

$$\frac{P_{XYST}(x,y,s,t)}{P(x,y)} = \frac{P_S(s)P_T(t)}{\sum_{s'\ni x} P_S(s') \sum_{t'\ni y} P_T(t')}$$

if $(x,y) \in s \times t$, and

$$\frac{P_{XYST}(x,y,s,t)}{P(x,y)} = 0$$

otherwise. Summing both sides over $t$, we see that $p(s|x,y) = p(s|x)$ for all $y$ such that $\{x,y\} \in E$, and

$$p(s|x) = \begin{cases} \frac{P_S(s)}{\sum_{s'\ni x} P_S(s')}, & \text{if } x \in s \\ 0, & \text{else.} \end{cases}$$

Thus, the first part of (53) is verified, and this is trivially completed by repeating the above analysis for $p(t|y)$. Now, (51) is easily verified by substitution. □

We next discuss the coincidence of $\mathcal{R}^{\text{in}}(G,P_X,P_Y)$ and $\mathcal{R}^{\text{out}}(G,P_X,P_Y)$. We will provide sufficient conditions in terms of the graph $G$ for this coincidence for some nonvanishing pair of marginals $P_X$ and $P_Y$. Toward that end, we first prove the following lemma.

*Lemma 3:* For all triplets $G$, $P_X$, and $P_Y$,

$$H(Y) + R_X^{\text{out}}(G_X,P_X) \geq \max_{\{X',Y'\}\in E, X'\sim P_X, Y'\sim P_Y} H(X',Y') \quad (56)$$

$$H(X) + R_Y^{\text{out}}(G_Y,P_Y) \geq \max_{\{X',Y'\}\in E, X'\sim P_X, Y'\sim P_Y} H(X',Y') \quad (57)$$

*Proof:* We only prove (56) and the proof of (57) similarly follows. Using (38), we can write

$$\begin{aligned} H(Y) + R_X^{\text{out}}(G_X,P_X) &= H(X) + H(Y) - H(\overline{G_X},P_X) \\ &= H(X) + H(Y) \\ &\quad - \min_{X\in S_X\in\mathcal{T}(G_X)} I(X;S_X). \end{aligned}$$

Also,

$$\max_{\{X,Y\}\in E, X\sim P_X, Y\sim P_Y} H(X,Y) = H(X)+H(Y) - \min_{\{X,Y\}\in E, X\sim P_X, Y\sim P_Y} I(X;Y).$$

Therefore, the result follows if

$$\min_{\{X,Y\}\in E, X\sim P_X, Y\sim P_Y} I(X;Y) \geq \min_{X\in S_X\in \mathcal{T}(G_X)} I(X;S_X) \tag{58}$$

holds. Let $P(x,y) = P_X(x)p(y|x)$ attain the left-hand side minimum in (58). Form the sets

$$s_X(y) = \{x : P(x,y) > 0\}$$

for each $y \in \mathcal{Y}$, and let

$$p(s_X(y)|x) = p(y|x)$$

so that $I(X;Y) = I(X;S_X)$. The result then follows by observing i) the sets $s_X(y)$ are complete subgraphs of $G_X$, and ii) the value of the minimum on the right-hand side in (58) does not change when the collection of valid sets $s_X$ is extended from $\mathcal{T}(G_X)$ to the set of *all* complete subgraphs. □

*Theorem 5:* Let $G = (\mathcal{X}\cup\mathcal{Y}, E)$ be an arbitrary bipartite graph. There exists some joint distribution $P(x,y)$ with nonvanishing marginals such that $G$ is the characteristic graph of $P$, and

$$\mathcal{R}^{\text{out}}(G, P_X, P_Y) = \mathcal{R}^{\text{in}}(G, P_X, P_Y) \tag{59}$$

if $G$ has two exact bipartite covers of the form

$$\mathcal{C}_1 = (\{\{x\} : x\in\mathcal{X}\}, \mathcal{C}_{Y,1})$$
$$\mathcal{C}_2 = (\mathcal{C}_{X,2}, \{\{y\} : y\in\mathcal{Y}\}) \tag{60}$$

and $|\mathcal{X}|\cdot|\mathcal{C}_{Y,1}| = |E| = |\mathcal{C}_{X,2}|\cdot|\mathcal{Y}|$ is satisfied.

*Remark:* For a general exact bipartite cover $\mathcal{C} = (\mathcal{C}_X, \mathcal{C}_Y)$, the relation $|\mathcal{C}_X|\cdot|\mathcal{C}_Y| = |E|$ holds if and only if the sets in $\mathcal{C}_X$ or $\mathcal{C}_Y$ do not overlap. See Fig. 10 for a demonstration of an exact bipartite cover of the form $\mathcal{C} = (\{\{x\} : x\in\mathcal{X}\}, \mathcal{C}_Y)$ for which $|\mathcal{X}|\cdot|\mathcal{C}_Y| > |E|$.

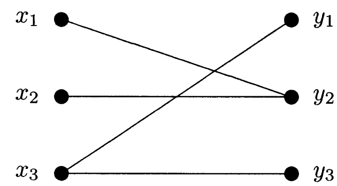



Fig. 10. The bipartite cover $\mathcal{C}_X = \{\{x_1\},\{x_2\},\{x_3\}\}$ and $\mathcal{C}_Y = \{\{y_1,y_2\},\{y_2,y_3\}\}$ is exact, but $|\mathcal{C}_X|\cdot|\mathcal{C}_Y| = 6 > 4 = |E|$.

*Proof:* From Theorem 4, it follows that if an exact bipartite cover of the form $\mathcal{C}_1 = (\{\{x\} : x\in\mathcal{X}\}, \mathcal{C}_{Y,1})$ exists, then using independent random variables $S$ and $T$, nonvanishing $P_X$ and $P_Y$ on $G$, with $X = S$, can be constructed such that

$$H(X) + I(Y;T) = \max_{\{X',Y'\}\in E, X'\sim P_X, Y'\sim P_Y} H(X',Y'). \tag{61}$$

Moreover, $(H(X), I(Y;T)) \in \mathcal{R}^{\text{in}}(G,P_X,P_Y)$. Now, using Lemma 3, we have

$$\begin{aligned} H(X) + I(Y;T) &\geq H(X) + R_Y^{\text{in}}(G_Y, P_Y) \\ &\geq H(X) + R_Y^{\text{out}}(G_Y, P_Y) \\ &\geq \max_{\{X',Y'\}\in E, X'\sim P_X, Y'\sim P_Y} H(X',Y'). \end{aligned}$$

Thus, (61) implies satisfaction of all inequalities above with equality. In particular, $(R_X, R_Y) \in \mathcal{R}^{\text{in}}(G, P_X, P_Y)$, where

$$\begin{aligned} R_Y &= R_Y^{\text{out}}(G_Y, P_Y) \\ R_X + R_Y &= R_T^{\text{out}}(G, P_X, P_Y). \end{aligned}$$

A similar result for the other "corner" of $\mathcal{R}^{\text{out}}(G,P_X,P_Y)$ is guaranteed by the existence of $\mathcal{C}_2 = (\mathcal{C}_{X,2}, \{\{y\} : y\in\mathcal{Y}\})$. However, by (55) in the proof of Theorem 4, the marginals $P_X$ and $P_Y$ on $G$ achieving the two corner points may be different. We will show that such a situation cannot arise if $\mathcal{C}_1$ and $\mathcal{C}_2$ further satisfy $|\mathcal{X}|\cdot|\mathcal{C}_{Y,1}| = |E| = |\mathcal{C}_{X,2}|\cdot|\mathcal{Y}|$. Note that, in this case, since every edge $\{x,y\} \in E$ may be indexed by a pair $(s,y) \in \mathcal{C}_{X,2}\times\mathcal{Y}$ (or $(x,t) \in \mathcal{X}\times\mathcal{C}_{Y,1}$), clearly, for distinct $s, s' \in \mathcal{C}_{X,2}$, we have $s\cap s' = \emptyset$ (and, for distinct $t, t' \in \mathcal{C}_{Y,1}, t\cap t' = \emptyset$).

Now, in the proof of Theorem 4, choose the distributions

$$P_{S,1}(s) = \frac{1}{|\mathcal{X}|} \quad\text{and}\quad P_{T,1}(t) = \frac{1}{|\mathcal{C}_{Y,1}|}$$

over $s\in\mathcal{X}$ and $t\in\mathcal{C}_{Y,1}$ for $\mathcal{C}_1 = (\mathcal{X}, \mathcal{C}_{Y,1})$, and

$$P_{S,2}(s) = \frac{1}{|\mathcal{C}_{X,2}|} \quad\text{and}\quad P_{T,2}(t) = \frac{1}{|\mathcal{Y}|}$$

over $s\in\mathcal{C}_{X,2}$ and $t\in\mathcal{Y}$ for $\mathcal{C}_2 = (\mathcal{C}_{X,2}, \mathcal{Y})$. If $P_1(x,y)$ and $P_2(x,y)$ are the distributions in (55) corresponding to $P_{S,1}(s)$ and $P_{T,1}(t)$, and $P_{S,2}(s)$ and $P_{T,2}(t)$, respectively, then we have

$$\begin{aligned} P_1(x,y) &= P_{S,1}(x)\cdot P_{T,1}(t) \\ &= \frac{1}{|\mathcal{X}|\cdot|\mathcal{C}_{Y,1}|} = \frac{1}{|E|} \\ &= \frac{1}{|\mathcal{C}_{X,2}|\cdot|\mathcal{Y}|} \\ &= P_{S,2}(s)\cdot P_{T,2}(y) \\ &= P_2(x,y) \end{aligned}$$

for $\{x,y\}\in E$, and $P_1(x,y) = P_2(x,y) = 0$ if $\{x,y\}\notin E$. □

Thus, Theorems 4 and 5 provide purely information-theoretic characterizations of purely combinatorial properties of bipartite graphs. Recalling the discussion in Section III-F about Question 1 of successive encoding, the concept of a bipartite graph with an exact bipartite cover may be understood as an extension of the concept of a normal graph.

As noted earlier, the question addressed in the following theorem is analogous to Question 2 of the successive encoding case. In answering this question in [9], Csiszár *et al.* made use of results from polyhedral combinatorics, and discovered a characterization of the important class of perfect graphs. But the next theorem shows that correspondingly requiring equality in (44) for *all* marginals $P_X$ and $P_Y$ on $G$ is too restrictive: a relatively simple proof shows that such a requirement is equivalent to restricting $G$ to the rather uninteresting class of disjoint collections of complete bipartite graphs.

*Theorem 6:* Let $G = (\mathcal{X} \cup \mathcal{Y}, E)$ be an arbitrary bipartite graph

$$\min_{\mathcal{C}} \min_{X \in S \in \mathcal{C}_X, Y \in T \in \mathcal{C}_Y} I(X;S) + I(Y;T) = \max_{\{X',Y'\} \in E, X' \sim P_X, Y' \sim P_Y} H(X',Y') \quad (62)$$

for every pair of marginals $P_X$ and $P_Y$ on $G$ if and only if $G$ is a disjoint collection of complete bipartite graphs.

*Proof:* For the "only if" part, suppose that $G$ is not a collection of disjoint complete bipartite graphs. Equivalently, $G$ has a connected component (i.e., a component wherein there is a path between any two nodes), denoted $G'$, which is not complete. The statement that $G'$ is not complete is equivalent to the statement: $G'$ has an induced Z-shaped bipartite subgraph of the form

$$G_S = (\mathcal{X}_S \cup \mathcal{Y}_S, E_S) = (\{x_1, x_2, y_1, y_2\}, \{\{x_1, y_1\}, \{x_2, y_1\}, \{x_2, y_2\}\}).$$

Let $P_X$ and $P_Y$ be any marginals such that $P_X(x) > 0$ and $P_Y(y) > 0$ if $x \in \mathcal{X}_S, y \in \mathcal{Y}_S$, and 0 otherwise. The only bipartite cover of $G_S$ is $(\{\{x_1\},\{x_2\}\}, \{\{y_1\},\{y_2\}\})$, so that

$$R_T^{\text{in}}(G, P_X, P_Y) = H(X) + H(Y).$$

Let $X'$ and $Y'$ be jointly distributed with $P(x,y)$ which attains $R_T^{\text{out}}(G, P_X, P_Y)$. Thus, $\{X', Y'\} \in E_S$, and $X' \sim P_X$, $Y' \sim P_Y$, and

$$\begin{aligned} R_T^{\text{in}}(G, P_X, P_Y) - R_T^{\text{out}}(G, P_X, P_Y) &= H(X) + H(Y) - H(X', Y') \\ &= H(X') + H(Y') - H(X', Y') \\ &= I(X'; Y'). \end{aligned}$$

But $I(X';Y') = 0$ if and only if $P(x,y) = P_X(x)P_Y(y)$ for all $x \in \mathcal{X}_S, y \in \mathcal{Y}_S$. This is impossible, since $P(x_1, y_2) = 0$ by construction, while $P_X(x_1)P_Y(y_2) \neq 0$.

Now for the "if" part. Let $G = (\mathcal{X} \cup \mathcal{Y}, E)$, with

$$\mathcal{X} \cup \mathcal{Y} = \bigcup_{k=1}^{K} (\mathcal{X}_k \cup \mathcal{Y}_k)$$
$$\mathcal{X}_k \cap \mathcal{X}_{k'} = \phi = \mathcal{Y}_k \cap \mathcal{Y}_{k'}$$

for $k \neq k'$, and $E = \bigcup_{k=1}^{K} E_k$, where

$$E_k = \{\{x,y\} : x \in \mathcal{X}_k, y \in \mathcal{Y}_k\}$$

be a collection of $K$ disjoint complete bipartite graphs, and let $(P_X, P_Y)$ be arbitrary marginals on $G$. Note that any such marginals satisfy

$$\sum_{x \in \mathcal{X}_k} P_X(x) = \sum_{y \in \mathcal{Y}_k} P_Y(y) \overset{\text{def}}{=} P_k$$

for every $k$. Define the auxiliary distribution $P_{X,k}$ by

$$P_{X,k}(x) = \begin{cases} \frac{P_X(x)}{P_k}, & \text{if } x \in \mathcal{X}_k \\ 0, & \text{else} \end{cases}$$

and define $P_{Y,k}$ similarly. As convenient, we will interchangingly denote entropy as function of its random variable or its distribution, e.g., $H(X)$ or $\mathcal{H}(P_X)$.

Let us begin by calculating $R_T^{\text{in}}(G, P_X, P_Y)$. Note that $G$ has the exact bipartite cover $\mathcal{C} = (\mathcal{C}_X, \mathcal{C}_Y)$, where $\mathcal{C}_Y = \{\{y\} : y \in \mathcal{Y}\}$, and the collection $\mathcal{C}_X$ is composed of all distinct sets of the form $\{x_1, x_2, \ldots, x_K\}$, where $x_k \in \mathcal{X}_k$. Set $T = Y$, and define $S$ by the distribution

$$p(s = \{x_1, x_2, \ldots, x_K\} | x) = \begin{cases} \prod_{j \neq k} P_{X,j}(x_j), & \text{if } x = x_k \\ 0, & \text{else.} \end{cases}$$

The corresponding marginal can be calculated as

$$P_S(s = \{x_1, x_2, \ldots, x_K\}) = \prod_{k=1}^{K} P_{X,k}(x_k).$$

Then $I(Y;T) = H(Y)$, and $I(X;S) = H(S) - H(S|X)$, where

$$H(S) = \mathcal{H}\left(\prod_{k=1}^{K} P_{X,k}(x_k)\right) = \sum_{k=1}^{K} \mathcal{H}(P_{X,k})$$

and

$$\begin{aligned} H(S|X) &= \sum_{k=1}^{K} \left(\sum_{x \in \mathcal{X}_k} P_X(x)\right) \sum_{j \neq k} \mathcal{H}(P_{X,j}) \\ &= \sum_{k=1}^{K} \mathcal{H}(P_{X,k})(1 - P_k) \end{aligned}$$

so that

$$\begin{aligned} R_T^{\text{in}}(G, P_X, P_Y) &\le I(X;S) + I(Y;T) \\ &= H(Y) + \sum_{k=1}^{K} \mathcal{H}(P_{X,k}) P_k. \end{aligned} \quad (63)$$

We now turn to $R_T^{\text{out}}(G, P_X, P_Y)$. Choose the random variables $(X', Y')$ according to the distribution

$$P(x,y) = \begin{cases} \frac{P_X(x)P_Y(y)}{P_k}, & \text{if } \{x,y\} \in E_k \text{ for some } k \\ 0, & \text{else.} \end{cases}$$

Note that $\{X', Y'\} \in E$, and $X' \sim P_X$, $Y' \sim P_Y$. Further, for any $y \in \mathcal{Y}_k$, $P(x|y) = P_{X,k}(x)$ if $x \in \mathcal{X}_k$, and 0 otherwise. Thus,

$$\begin{aligned} H(X'|Y') &= \sum_{k=1}^{K} \sum_{y \in \mathcal{Y}_k} P_Y(y) H(X|Y=y) \\ &= \sum_{k=1}^{K} \mathcal{H}(P_{X,k}) P_k. \end{aligned}$$

Since

$$\begin{aligned} R_T^{\text{out}}(G, P_X, P_Y) \ge H(X', Y') &= H(Y') + H(X'|Y') \\ &= H(Y) + H(X'|Y') \end{aligned}$$

a comparison with (63) shows that

$$R_T^{\text{in}}(G, P_X, P_Y) = R_T^{\text{out}}(G, P_X, P_Y). \qquad \square$$

It may be observed that the choices made in the proof for the bipartite covers and random variables correspond to the following successive encoding strategy, which is intuitively obvious: Bob directly encodes $Y$, expending a rate of $H(Y)$. Merlin now knows $G_k$ to which $(x,y)$ belongs; Alice identifies $X$ within $\mathcal{X}_k$, expending an expected rate of $\sum_{k=1}^{K} \mathcal{H}(P_{X,k})P_k$.

*Examples:*

1) For the example in Section III-E, we already know that there exists a pair of nonvanishing marginals on $G$ so that (44) holds with equality. Therefore, $G$ must have an exact bipartite cover. In fact, the bipartite cover given by

$$\mathcal{C}_X = \{\{x_1, x_3\}, \{x_2\}, \{x_4\}\}$$

and

$$\mathcal{C}_Y = \{\{y_1, y_4\}, \{y_2, y_3\}\}$$

which was used for computation of the point $(\log 3, 1)$, is exact. Further, $P_X$ and $P_Y$ in the example are induced by $P(x,y)$ as in (55) with $P_S(s) = \frac{1}{3}$ and $P_T(t) = \frac{1}{2}$.

2) Suppose

$$G_{2a} = (\{0, 1, \ldots, 2a-1\} \cup \{0, 1, \ldots, 2a-1\}, E_{2a})$$

where

$$E_{2a} = \{\{x, y\} : y = x \text{ or } y = x + 1 \bmod 2a\}.$$

(Thus, $G_{2a}$ is the Shannon typewriter channel on $2a$ letters.) Retain the notation of Theorem 5. Clearly

$$\mathcal{C}_1 = \left(\left\{\{x\} : 0 \le x \le 2a-1\right\}, \left\{\{0, 2, \ldots, 2a-2\}, \{1, 3, \ldots, 2a-1\}\right\}\right)$$

$$\mathcal{C}_2 = \left(\left\{\{0, 2, \ldots, 2a-2\}, \{1, 3, \ldots, 2a-1\}\right\}, \left\{\{y\}: 0 \le y \le 2a-1\right\}\right)$$

are exact bipartite covers of $G_{2a}$, and they satisfy $|\mathcal{X}| \cdot |\mathcal{C}_{Y,1}| = |E| = 4a = |\mathcal{C}_{X,2}| \cdot |\mathcal{Y}|$. Thus, equality holds in (59) for every $G_{2a}$, $a \ge 1$, with $P_X$ and $P_Y$ being the corresponding uniform distributions.

## V. Bounds for Finite Block Length

Bounds for achievable rates for a finite and fixed block length are of interest from the algorithmic perspective, as well as in the study of rates of convergence to the asymptotic limits discussed in the previous sections.

Let us denote by $\mathcal{R}_n^{\text{pf}}(G, P_X, P_Y)$, $\mathcal{R}_n^{\text{inst}}(G, P_X, P_Y)$, and $\mathcal{R}_n^{\text{ud}}(G, P_X, P_Y)$ the achievable rate regions for block length $n$ for the respective variable-length coding classes. From the definition of those classes, it is clear that

$$\mathcal{R}_n^{\text{pf}}(G, P_X, P_Y) \subseteq \mathcal{R}_n^{\text{inst}}(G, P_X, P_Y) \subseteq \mathcal{R}_n^{\text{ud}}(G, P_X, P_Y)$$

for all $G$, $P_X$, and $P_Y$. Unlike their asymptotic counterparts, these regions do not necessarily coincide.

It follows from the definition of prefix-free codes that

$$\mathcal{R}_n^{\text{pf}}(G, P_X, P_Y) \supseteq \bigcup_{\Phi_X, \Phi_Y} \left(\frac{1}{n}\left[H(\Phi_X(X_1^n)) + 1\right], \frac{1}{n}\left[H(\Phi_Y(Y_1^n)) + 1\right]\right) \quad (64)$$

where the union is over all bipartite colorings of $G^n$. Also, recall from the proof of Theorem 1 that if $(R_X, R_Y) \in \mathcal{R}_n^{\text{ud}}(G, P_X, P_Y)$, then (11) holds. Therefore,

$$\mathcal{R}_n^{\text{ud}}(G, P_X, P_Y) \subseteq \bigcap_{0 \le \alpha \le 1} \left\{(R_X, R_Y) : \alpha R_X + (1-\alpha) R_Y \ge \frac{1}{n} H_\chi(G^n, P_X^n, P_Y^n, \alpha) - r(n)\right\}. \quad (65)$$

Next, we turn our attention to the class of scalar instantaneous codes. Naturally, (65) also induces an outer bound for $\mathcal{R}_1^{\text{inst}}(G, P_X, P_Y)$. In the next theorem, we derive an alternate outer bound which is sometimes tighter.

*Theorem 7:*

$$\mathcal{R}_1^{\text{inst}}(G, P_X, P_Y) \subseteq \mathcal{R}^{\text{in}}(G, P_X, P_Y). \quad (66)$$

*Proof:* The proof is modeled on [1, proof of Theorem 4]. Given a scalar instantaneous code $(\phi_X, \phi_Y, \psi)$ for $G$ with rates

$$\bar{l}(\phi_X) \stackrel{\text{def}}{=} \sum_{x \in \mathcal{X}} P_X(x) |\phi_X(x)|$$
$$\bar{l}(\phi_Y) \stackrel{\text{def}}{=} \sum_{y \in \mathcal{Y}} P_Y(y) |\phi_Y(y)|.$$

We will construct random variables $S$ and $T$ such that $X \in S \in \mathcal{C}_X$ and $Y \in T \in \mathcal{C}_Y$ for some bipartite cover $\mathcal{C} = (\mathcal{C}_X, \mathcal{C}_Y)$, and

$$\bar{l}(\phi_X) \ge I(X; S)$$
$$\bar{l}(\phi_Y) \ge I(Y; T).$$

The theorem will then follow from (25).

Let $T^{\phi_X}$ be the (binary) tree whose vertices are all the strings in $\{\phi_X(x) : x \in \mathcal{X}\}$ and their prefixes. Similarly define $T^{\phi_Y}$. We may assume that neither $T^{\phi_X}$ nor $T^{\phi_Y}$ contains a vertex with a single descendant. (Otherwise, the corresponding tree can be pruned, thus reducing the rate sum.) Associate with every vertex $\tilde{x}$ of $T^{\phi_X}$ the set $\phi_X^{-1}(\tilde{x}) = \{x : \phi_X(x) = \tilde{x}\}$. Similarly define $\phi_Y^{-1}(\tilde{y})$ for vertices $\tilde{y} \in T^{\phi_Y}$. Note that $\phi_X^{-1}(\tilde{x})$ is never empty when $\tilde{x}$ is a leaf, but may be empty for internal vertices (and similarly for $\tilde{y}$).

Associate with each leaf $\tilde{s}$ of $T^{\phi_X}$ and $\tilde{t}$ of $T^{\phi_Y}$ the respective sets

$$s = \bigcup_{\tilde{x} \text{ prefixes } \tilde{s}} \phi_X^{-1}(\tilde{x})$$
$$t = \bigcup_{\tilde{y} \text{ prefixes } \tilde{t}} \phi_X^{-1}(\tilde{y}).$$

Denote the collections of all such $s$ and $t$ by $\mathcal{C}_X$ and $\mathcal{C}_Y$, respectively. Every $x \in \mathcal{X}$ (and $y \in \mathcal{Y}$) is contained in some $s \in \mathcal{C}_X$ (some $t \in \mathcal{C}_Y$). Further, since $(\phi_X, \phi_Y, \psi)$ is an instantaneous code, by (2) there exists at most one edge $\{x, y\} \in E$ in the set

$s \times t$, for any $s \in \mathcal{C}_X$ and $t \in \mathcal{C}_Y$. Thus, $(\mathcal{C}_X, \mathcal{C}_Y)$ is a bipartite cover of $G$.

Let $leaves\,(\tilde{x})$ be the set of leaves which descend from a vertex $\tilde{x} \in T^{\phi_X}$. Similarly define $leaves\,(\tilde{y})$ for vertices $\tilde{y} \in T^{\phi_Y}$. Since $T^{\phi_X}$ and $T^{\phi_Y}$ have no vertices with single descendants

$$\sum_{\tilde{s} \in leaves\,(\tilde{x})} 2^{-(|\tilde{s}|-|\tilde{x}|)} = 1$$
$$\sum_{\tilde{t} \in leaves\,(\tilde{y})} 2^{-\left(|\tilde{t}|-|\tilde{y}|\right)} = 1$$

for every $\tilde{x} \in T^{\phi_X}$ and every $\tilde{y} \in T^{\phi_Y}$. So $p(\tilde{s}|\tilde{x}) = 2^{-(|\tilde{s}|-|\tilde{x}|)}$ for $\tilde{s} \in leaves\,(\tilde{x})$, and $p(\tilde{t}|\tilde{y}) = 2^{-\left(|\tilde{t}|-|\tilde{y}|\right)}$ for $\tilde{t} \in leaves\,(\tilde{y})$, are probability distributions on $leaves\,(\tilde{x})$ and $leaves\,(\tilde{y})$, respectively.

Define the random variables $S$ and $T$ over $\mathcal{C}_X$ and $\mathcal{C}_Y$ by $p(s|x) = p(\tilde{s}|\tilde{x})$ if $\tilde{s} \in leaves\,(\tilde{x} = \phi_X(x))$, and $p(s|x) = 0$ otherwise. Define $p(t|y)$ similarly. Then $X \in S \in \mathcal{C}_X$ and $Y \in T \in \mathcal{C}_Y$. Furthermore, the maps $s \mapsto \tilde{s}$ and $t \mapsto \tilde{t}$ are prefix-free codes for $S$ and $T$, respectively. Therefore,

$$\begin{aligned} H(S) \leq \sum_{s \in \mathcal{C}_X} p(s)|\tilde{s}| &= \sum_{x \in \mathcal{X}} P_X(x) \sum_{s \in \mathcal{C}_X} p(s|x)|\tilde{s}| \\ &= \sum_{x \in \mathcal{X}} P_X(x) \sum_{s \in \mathcal{C}_X} p(s|x)\left\{|\tilde{x}| + |\tilde{s}| - |\tilde{x}|\right\} \\ &= \bar{l}(\phi_X) - \sum_{x \in \mathcal{X}} P_X(x) \sum_{s \in \mathcal{C}_X} p(s|x) \log p(s|x) \\ &= \bar{l}(\phi_X) + H(S|X) \end{aligned}$$

and hence $\bar{l}(\phi_X) \geq I(X;S)$. It can be similarly shown that $\bar{l}(\phi_Y) \geq I(Y;T)$, and the theorem follows. □

We close this section with an example showing that (66) can indeed be tighter than (65). Let $G$ be a complete bipartite graph. Then setting $\alpha = 1/2$ and $n = 1$ in (65), we get

$$\begin{aligned} R_X + R_Y \geq\, & H(X) + H(Y) - \log(1 + \log|\mathcal{X}|) \\ & - \log(1 + \log|\mathcal{Y}|) - 2\log e. \end{aligned}$$

On the other hand, (66) gives a tighter bound

$$R_X + R_Y \geq H(X) + H(Y).$$

## VI. Asymptotics of Fixed-Length Coding

In order to obtain an inner bound on $\mathcal{R}^{\text{fl}}(G)$ we appeal to the idea that yielded in Section III-C an inner bound for the asymptotically achievable expected rate region. That is, we observe that fixed-length codes satisfy the necessary condition (23). This condition is weaker than (24), which is necessarily satisfied by fixed-length codes which enable Merlin to evaluate the function $e$ of (22). Thus, if $\mathcal{R}^{\text{fl,in}}(G)$ denotes the asymptotically achievable rate region of such codes, we obtain that $\mathcal{R}^{\text{fl,in}}(G) \subseteq \mathcal{R}^{\text{fl}}(G)$.

Now for an outer bound. If $(\phi_X, \phi_Y, \psi)$, with $\phi_X : \mathcal{X}^n \to \{1, 2, \ldots, N_X\}$ and $\phi_Y : \mathcal{Y}^n \to \{1, 2, \ldots, N_Y\}$ is a code satisfying (23), then $(\phi_X(x^n), \phi_Y(y^n))$ takes a different value for each edge $\{x^n, y^n\} \in E_n$. Thus, $N_X N_Y \geq |E_n|$. But, by definition of $E_n$ in (1), $|E_n| = |E|^n$, and we have

$$R_X + R_Y \geq \log|E| \tag{67}$$

for all fixed-length codes $(\phi_X, \phi_Y, \psi)$. Also, even when Bob directly sends his information expending rate $\log|\mathcal{X}|$, the minimum rate Alice can achieve, by coloring the "side information" graph $G_X$, is given by $\frac{1}{n}\log\chi(G_X^n)$. Since there is no known single-letter formula for the limit of $\frac{1}{n}\log\chi(G_X^n)$, we further lower-bound Alice's rate using

$$\begin{aligned} \frac{1}{n}\log\chi(G_X^n) \geq & \frac{1}{n}\log\omega(G_X^n) \\ &= \frac{1}{n}\log\omega(G_X)^n \\ &= \log\omega(G_X)\,. \end{aligned}$$

Thus,

$$R_X \geq \log\omega(G_X) \tag{68}$$
$$R_Y \geq \log\omega(G_Y). \tag{69}$$

We define $\mathcal{R}^{\text{fl,out}}(G) \supseteq \mathcal{R}^{\text{fl}}(G)$ by bringing together (67), (68), and (69).

Notice that $\mathcal{R}^{\text{fl,out}}(G)$ can also be obtained by

$$\mathcal{R}^{\text{fl,out}}(G) = \bigcap_{P_X, P_Y \text{on } G} \mathcal{R}^{\text{out}}(G, P_X, P_Y)$$

since fixed-length codes cannot outperform variable-length codes designed for any pair of marginals $P_X$ and $P_Y$ on $G$. Using the same argument, we now derive a single-letter formula for $\mathcal{R}^{\text{fl,in}}(G)$.

*Theorem 8:*

$$\mathcal{R}^{\text{fl,in}}(G) = \bigcap_{P_X, P_Y \text{on } G} \mathcal{R}^{\text{in}}(G, P_X, P_Y)\,. \tag{70}$$

*Proof:* Suppose Alice and Bob must design variable-length codes satisfying (24) for unknown marginals $P_X$ and $P_Y$ on a given characteristic graph $G$ (e.g., the marginals are chosen by an adversary). Clearly, the achieved rate pair $(R_X, R_Y)$ satisfies $(R_X, R_Y) \in \mathcal{R}^{\text{in}}(G, P_X, P_Y)$ for every pair of marginals $P_X$ and $P_Y$ on $G$. On the other hand, since a possible coding strategy for Alice and Bob is to use fixed-length codes, we have

$$\mathcal{R}^{\text{fl,in}}(G) \subseteq \bigcap_{P_X, P_Y \text{on } G} \mathcal{R}^{\text{in}}(G, P_X, P_Y).$$

Now for the reverse direction. The *type* of $x_1^n \in \mathcal{X}^n$ is the distribution $P_X$ on $\mathcal{X}$ defined by

$$P_X(a) = \frac{1}{n}N(a|x^n) \text{ for every } a \in \mathcal{X}.$$

Recall the definition of typical sets from (18). If the pair of types $P_X$ and $P_Y$ are marginals on $G$, the proof of Theorem 2 shows that, for any $(R_X, R_Y) \in \mathcal{R}^{\text{in}}(G, P_X, P_Y)$, and for large enough $n$, there exists a code $(\phi_X, \phi_Y, \psi)$, where $\phi_X : \mathcal{X}^n \to \{1, 2, \ldots, N_X\}$ and $\phi_Y : \mathcal{Y}^n \to \{1, 2, \ldots, N_Y\}$ such that $N_X \leq 2^{nR_X}$ and $N_Y \leq 2^{nR_Y}$, which satisfies (24) for any $(x_1^n, y_1^n)$ and $(x_1'^n, y_1'^n)$ with $x_1^n, x_1'^n \in T^n_{[P_X]_\epsilon}$ and $y_1^n, y_1'^n \in T^n_{[P_Y]_\epsilon}$. A fixed-length code for the entire space $\mathcal{X}^n \times \mathcal{Y}^n$ is now obtained by choosing such a code for each pair of types $P_X$ and $P_Y$, and preceding the codewords for each pair of types with a pair of type-specifying indexes. The claim follows from the fact that the number of distinct pairs of types $(P_X, P_Y)$ grows only polynomially with $n$, and hence, the additional rate expended for such indices vanishes as $n \to \infty$. □

Define

$$R_T^{\mathrm{fl,in}}(G) \stackrel{\mathrm{def}}{=} \min_{(R_X, R_Y) \in \mathcal{R}^{\mathrm{fl,in}}(G)} R_X + R_Y.$$

It then follows from (70) and (42) that

$$\begin{aligned} &R_T^{\mathrm{fl,in}}(G) \\ &\quad = \max_{P_X, P_Y \text{ on } G} R_T^{\mathrm{in}}(G, P_X, P_Y) \\ &\quad = \max_{P_X, P_Y \text{on } G} \min_{\mathcal{C}} \min_{X \in S \in \mathcal{C}_X, Y \in T \in \mathcal{C}_Y} I(X;S) + I(Y;T). \end{aligned}$$

We next consider a simple $R_T^{\mathrm{fl,in}}(G)$ calculation example. Let $G$ be as in Fig. 8, and fix $\mathcal{C} = (\mathcal{C}_X, \mathcal{C}_Y)$, where

$$\mathcal{C}_X = \{\{x_1, x_3\}, \{x_2\}, \{x_4\}\}$$

and

$$\mathcal{C}_Y = \{\{y_1, y_4\}, \{y_2, y_3\}\}.$$

Then

$$R_T^{\mathrm{fl,in}}(G) \leq \max_{P_X, P_Y \text{on } G} \min_{X \in S \in \mathcal{C}_X, Y \in T \in \mathcal{C}_Y} I(X;S) + I(Y;T).$$

By an application of [20, Theorem 2], the expression on the right-hand side is seen to be $\log 6$; this is achieved by choosing $(P_X, P_Y)$ as in Fig. 8, and choosing, for $s \in \mathcal{C}_X$, $t \in \mathcal{C}_Y$, $p(s|x) = p(t|y) = 1$ if and only if $x \in s, y \in t$. Since $\log |E| = \log 6$ as well, we conclude from (67) that $R_T^{\mathrm{fl,in}}(G) = \log 6$. Further, it is easily seen that $\frac{1}{n} \log N_X + \frac{1}{n} \log N_Y = \log 6$ is achieved by a scalar code $(\phi_X, \phi_Y, \psi)$ setting

$$\phi_X(x_1) = \phi_X(x_3) \neq \phi_X(x_2) \neq \phi_X(x_4)$$

and

$$\phi_Y(y_1) = \phi_Y(y_4) \neq \phi_Y(y_2) = \phi_Y(y_3).$$

## VII. Conclusion

We initiated the study of rates of *zero-error* transmission for two senders who wish to convey correlated information from their respective sources to a receiver, when no communication is permitted between them. While a single-letter formula for the asymptotically achievable rate pairs remains elusive, we derived single-letter inner and outer bounds for both fixed- and variable-length coding. These bounds specialize to known results for the unknown side information problem, where one sender directly conveys his/her information expending full rate. Depending on circumstances, the inner/outer bounds can vary from tight to involving an arbitrarily large gap. We analyzed conditions for tightness in terms of purely combinatorial properties of the underlying characteristic graph. We also showed, via an example, that successive encoding combined with time sharing does not span the entire achievable rate region, in contrast with the Slepian–Wolf setup, where an asymptotically vanishing probability of error is tolerated. Finally, we derived bounds for variable-length coding with a finite block length, and for fixed-length coding with infinite block length.

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