On the Extreme Cases of the Rate-Distortion Function for Robust Descriptions

Ertem Tuncel and Kenneth Rose
Dept. of Electrical and Computer Engineering, University of California, Santa Barbara
{ertem,rose}@ece.ucsb.edu

We consider a multi-terminal source coding scenario known as *robust descriptions* coding [1, Section VII], where the source is to be encoded in a way that enables good descriptions for several distortion measures simultaneously. Multiple decoders, reconstructing the source according to their respective distortion measures, may represent users with different perception systems (e.g., humans vs. machines), or different quality expectations. Alternatively, they may represent a single user decoding the same bitstream for different applications (see e.g., [3] for the use of different distortion measures for searching and for reconstruction). The rate-distortion function for robust descriptions, derived in [1], is given by

$$R(D_1,\ldots,D_N) = \min_{E\{d_i(X,Y_i)\}\le D_i\ \forall i} I(X;Y_1,\ldots,Y_N)\ . \quad (1)$$

By trivial source coding arguments, we observe

$$\max_i R_i(D_i) \le R(D_1,\ldots,D_N) \le \sum_i R_i(D_i)\ , \quad (2)$$

where $R_i(\cdot)$ denotes the standard rate-distortion function for the measure $d_i(\cdot,\cdot)$. When the upper bound in (2) is tight, then the simple strategy of encoding the source separately for each distortion measure, and multiplexing or concatenating the descriptions is asymptotically optimal. The other extreme, i.e., achieving the lower bound in (2), is the best-case scenario, because then the encoder need not expend any more rate than the maximum of rates necessary to achieve the distortion levels $D_1,\ldots,D_N$ by individual encoding.

An equivalent characterization for $R(D_1,\ldots,D_N)$ is

$$L = \min_{Q_{y_1,\ldots,y_N|x}} I(X;Y_1,\ldots,Y_N) + \sum_i \beta_i E\, d_i(X,Y_i)\ , \quad (3)$$

where $\beta_1,\ldots,\beta_N$ are appropriate non-negative Lagrangian multipliers. Denote by $Q^*_{y_1,\ldots,y_N|x}$ any conditional distribution achieving the minimum in (3), and let $q^*_{y_1,\ldots,y_N} = \sum_x p_x Q^*_{y_1,\ldots,y_N|x}$. Our first result is obtained by posing robust descriptions as a special case of successive refinement coding, for which we obtained optimality conditions in [2].

Theorem 1: An optimal reproduction $q^*_{y_1,\ldots,y_N}$ satisfies

$$v(y_1,\ldots,y_N) \triangleq \sum_x \frac{p_x e^{-\sum_i \beta_i d_i(x,y_i)}}{\sum_{z_1,\ldots,z_N} q^*_{z_1,\ldots,z_N} e^{-\sum_i \beta_i d_i(x,z_i)}} \le 1 \quad (4)$$

and the corresponding conditional $Q^*_{y_1,\ldots,y_N|x}$ is given by

$$Q^*_{y_1,\ldots,y_N|x} = \frac{q^*_{y_1,\ldots,y_N} e^{-\sum_i \beta_i d_i(x,y_i)}}{\sum_{z_1,\ldots,z_N} q^*_{z_1,\ldots,z_N} e^{-\sum_i \beta_i d_i(x,z_i)}}\ . \quad (5)$$

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We next derive conditions for equality on either bound of (2), and show by examples that they can indeed occur.

Theorem 2: $R(D_1,\ldots,D_N) = \sum_i R_i(D_i)$ iff

$$q^*_{y_1,\ldots,y_N} = q^*_{y_1} q^*_{y_2} \cdots q^*_{y_N}\ . \quad (6)$$

Example: Let $N = 2$, $\mathcal{X} = \{0,1,2,3\}$, $\mathcal{Y}_1 = \mathcal{Y}_2 = \{a,b\}$, and $p_x = 1/4$ for all $x \in \mathcal{X}$. Also let

$d_1(x,y) =$

	a	b
0	0	1
1	1	0
2	1	0
3	0	1

and $d_2(x,y) =$

	a	b
0	0	1
1	0	1
2	1	0
3	1	0

This setup arises naturally in database searching where the users are only interested in identifying objects in a database that are most similar to a query object (see [3]). It can easily be verified using (4) and (6) with $q^*_{y_1} = q^*_{y_2} = 1/2$ that $R(D_1,D_2) = R_1(D_1) + R_2(D_2)$ for all D_1, D_2.

Now for the lower bound in (2). Without loss of generality, we assume that $R_1(D_1)$ is the largest among $R_i(D_i)$.

Theorem 3: $R(D_1,\ldots,D_N) = R_1(D_1)$ if and only if

$$\sum_{y_1} \min_{y_i\in\mathcal{Y}_i} \sum_x p_x P^*_{y_1|x} d_i(x,y_i) \le D_i\ , \quad (7)$$

for $2 \le i \le N$, where $P^*_{y_1|x}$ is an optimal conditional distribution attaining $R_1(D_1)$.

Example: Let $N = 2$, $\mathcal{X} = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0,1,2\}$, and $p_x = 1/3$ for all $x \in \mathcal{X}$. Also let $d_1(x,y_1) = |x - y_1|$ and $d_2(x,y_2) = d_H(x,y_2)$, i.e., the Hamming distance. Figure 1 below shows the relevant regions on the (D_1,D_2)-plane.

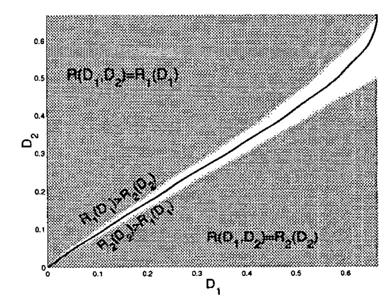

Figure 1: The gray color indicates the region where $R(D_1,D_2) = \max\{R_1(D_1), R_2(D_2)\}$, and the bold curve separates $R_1(D_1) > R_2(D_2)$ from $R_2(D_2) > R_1(D_1)$.

References

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[3] E. Tuncel and K. Rose. Rate-distortion approach to databases: storage and content-based retrieval. In *IEEE Int. Symp. on Information Theory*, page 348, Lausanne, Switzerland, July 2002.