The Zero-Error Capacity of Compound Channels

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The compound channel is one of the simplest generalizations of the discrete memoryless channel (DMC). A compound channel $\mathfrak{C}$ is defined by a set of DMC $\{p(x|y,s) : x \in \mathcal{X}, y \in \mathcal{Y}, s \in \mathcal{S}\}$, that share their input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$. $\mathcal{S}$ is an index set. All sets are assumed finite. Before transmission an arbitrary DMC s is chosen and the channel behaves like the chosen DMC for the duration of transmission. *We characterize the zero-error (as opposed to the more conventional requirement of asymptotically vanishing error) capacity when neither the encoder nor the decoder is informed of the choice of DMC s. We also study the effect of informing either the encoder or decoder of the choice of s.*

A study of the zero-error capacity of the compound channel was initiated by Cohen et al. [1] motivated by certain combinatorial problems. They presented an upper bound on the capacity in terms of certain graphs associated with the channel. In a remarkable paper, Gargano et al., [2] showed the bound is in fact tight. Our paper is motivated by the observation that the expression for the capacity presented in the above papers is accurate only in the case where the decoder is informed of s. However the other, central results of these papers remain valid.

We present a complete characterization of the compound channel capacities. But first, some notation:

For any input letter x and DMC s, the fan-out set $F_s(x)$ is the set $\{y \in \mathcal{Y} : p(y|x,s) > 0\}$. The fan-out set of a vector $\mathbf{x} \in \mathcal{X}^n$ with respect to the the channel s is the Cartesian product $F_s(\mathbf{x}) \triangleq F_s(x_1) \times \cdots \times F_s(x_n)$ and the fan-out with resect to the compound channel is $F_{\mathfrak{C}}(\mathbf{x}) \triangleq \cup_{s\in\mathcal{S}} F_s(\mathbf{x})$. Fan-out sets characterize the possible channel outputs for a given channel input. The zero-error requirement implies that any pair of elements in a valid code have non-intersecting fan-out sets. We define the characteristic set of graphs of a compound channel $\mathcal{G}_{\mathfrak{C}} = \{G_{ss'} : s, s' \in \mathcal{S}\}$ as follows: $G_{ss'} = (\mathcal{X}, E_{ss'})$, where $(x, x') \in E_{ss'} \Leftrightarrow F_s(x) \cap F_{s'}(x') = \emptyset$. Note that these graphs are directed in general. Also the graph can have self loops and $(v, v') \in E$ does not preclude $(v', v) \in E$. G_{ss} corresponds to the usual (undirected) characteristic graph of $\tilde{G}_s$ channel s [2]. Let $\mathcal{G}_{\mathcal{S}} \triangleq \{\tilde{G}_s : s \in \mathcal{S}\}$.

Given an undirected graph $G = (V, E)$, two vectors $\mathbf{x}, \mathbf{x}' \in V^n$ are incomparable if $\{x_i, x'_i\} \in E$ for some $1 \le i \le n$. If $N(G, n)$ denotes the cardinality of the largest set whose elements are pairwise incomparable, the Shannon capacity of the graph $C(G) \triangleq \lim_{n\to\infty} \frac{1}{n} \log N(G, n)$. Shannon showed that $C(G)$ is in fact the zero-error capacity of the channel with characteristic graph G [3]. The concept of capacity was extended to sets of graphs in [1]: if $N(\mathcal{G}, n)$ denotes the cardinality of the largest set whose elements are pairwise incomparable with respect to every graph in the set $\mathcal{G}$, the Shannon capacity of the set of graphs $C(\mathcal{G}) \triangleq \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{G}, n)$. Further generalizations of capacity were defined in [2] by appropriately defining incomparability with respect to directed graphs: two vectors $\mathbf{x}, \mathbf{x}' \in V^n$ are incomparable with respect to a directed graph $G = (V, E)$ if there exist co-ordinates i, j such that $(x_i, x'_i) \in E$ and $(x'_j, x_j) \in E$. Using this notion of incomparability, we can define the Sperner capacity of a directed graph G, denoted $\Sigma(G)$, and a set of directed graphs $\mathcal{G}$, denoted $\Sigma(\mathcal{G})$ as above. In [2], Gargano et al. obtained expressions for the Shannon and Sperner capacities of sets of graphs.

Our main result is

Theorem 1 *Given a compound channel $\mathfrak{C}$ with characteristic set of graphs $\mathcal{G}_{\mathfrak{C}}$ and set of characteristic graphs $\mathcal{G}_{\mathcal{S}}$, the capacities of $\mathfrak{C}$ are as follows–*

a) when neither encoder nor decoder is informed:

$$C^0(\mathfrak{C}) = \Sigma(\mathcal{G}_{\mathfrak{C}}).$$

b) when decoder is informed:

$$C^0_{dec}(\mathfrak{C}) = C(\mathcal{G}_{\mathcal{S}}).$$

c) when encoder is informed:

$$C^0_{enc}(\mathfrak{C}) = \begin{cases} 0 & \text{if } C^0(\mathfrak{C}) = 0 \\ \min_{s\in\mathcal{S}} C(G_s) & \text{otherwise} \end{cases}.$$

(a) follows from the definition of Sperner capacity and the zero-error requirement. (b) is essentially a careful restatement of the result in [1]. (c) relies on our result that if $\Sigma(\mathcal{G})$ is zero, then one of the graphs in $\mathcal{G}$ is edge-free.

These capacities are qualitatively different from those obtained for the conventional asymptotically vanishing probability of error case: in the conventional case we have $C(\mathfrak{C}) = C_{dec}(\mathfrak{C}) \le C_{enc}(\mathfrak{C})$ [4], while in the zero-error case we only have $C^0(\mathfrak{C}) \le C^0_{dec}(\mathfrak{C})$ and $C^0(\mathfrak{C}) \le C^0_{enc}(\mathfrak{C})$ in general. The differences arise because in the conventional case, the decoder can with high probability identify the DMC in operation purely by observing the channel output. However, this imperfect knowledge about the channel cannot be used for zero-error decoding.

Acknowledgments

Many thanks to E. Tuncel for some interesting discussions that led to the germination of this line of research.

References

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[1]This work was supported in part by the NSF under grants EIA-9986057 and EIA-0080134, the UC MICRO program, Dolby Laboratories, Inc., Lucent Technologies, Inc., Microsoft Corporation, Mindspeed Technologies, and Qualcomm, Inc.

0-7803-8280-3/04/$20.00 ©2004 IEEE