On Hierarchical Type Covering

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***Abstract*—This correspondence focuses on a significant distinction between two hierarchical type covering strategies, namely, *weak* and *strong* covering, and on the impact of this distinction on known results. In particular, it is demonstrated that the rate region for weak covering, whose natural use is in scalable source coding, is generally larger than the rate region for strong covering, which is primarily useful in hierarchical guessing. This correspondence also presents a corrected converse result for the hierarchical guessing problem.**

***Index Terms*—Hierarchical guessing, hierarchical type covering, scalable source coding.**

I. Introduction

Central to many rate-distortion theoretic results and proofs is the concept of *type covering*, i.e., covering of the set of source vectors with the same empirical distribution (type class) P using identical "spheres" that are centered at codevectors. For example, the achievability of Shannon's rate-distortion function $R_P(D)$ can be proven directly using the type covering lemma [4, Lemma 2.4.1], which asserts that for large n it suffices to use $\approx\ e^{nR_P(D)}$ spheres with radius D to entirely cover a type class P of length-n sequences. Here, the sphere of radius D about a codevector $\boldsymbol{y}$ is defined as the collection of source vectors $\boldsymbol{x}$ whose normalized distortion with $\boldsymbol{y}$ is less than or equal to D. In [16], the type covering lemma of [4] is refined in order to obtain an upper bound on the asymptotic *rate redundancy* of D-semifaithful codes. (D-semifaithful codes are variable-length codes that cover the entire source space and rate redundancy is defined as the difference between the minimum achievable average codeword length and the rate-distortion function $R_P(D)$). Another example is the achievability of optimum error exponents in universal lossy source coding [11]: Given a quota of total rate R, it suffices to cover those types $\hat{P}$ for which $R_{\hat{P}}(D) < R$ to make sure that the probability of error $\Pr[d(\boldsymbol{X},\boldsymbol{Y}) > D]$ decays exponentially fast in block length n with the optimum normalized exponent $E_P(D,R)$ for all sources P.

More recently, type covering was used in a related but different application, namely, guessing subject to distortion [1]: Bob draws a sample $\boldsymbol{x}$ from a discrete memoryless source (DMS), and Alice presents him with a fixed sequence of guesses $\boldsymbol{y}(1), \boldsymbol{y}(2), \ldots$, until Bob informs her that $d(\boldsymbol{x}, \boldsymbol{y}(i)) \leq D$. The objective is to design an optimal guessing strategy that minimizes the ρth moment of number of guesses $E\{G(\boldsymbol{X})^\rho\}$. The guessing game models a naive approach to quantizer codebook search where, given a data vector to be represented, codebook elements are sequentially read until a D-match is found. The guessing game can also be utilized for a similarity search scenario where a high-dimensional database is sequentially searched for a single D-matching vector to the query provided by the user. It turns out that the optimal strategy is to sort type classes P in increasing order of $R_P(D)$, and create the guessing list as the concatenation of codebooks (i.e., sets of codevectors) that cover type classes P_1, P_2, P_3, etc.

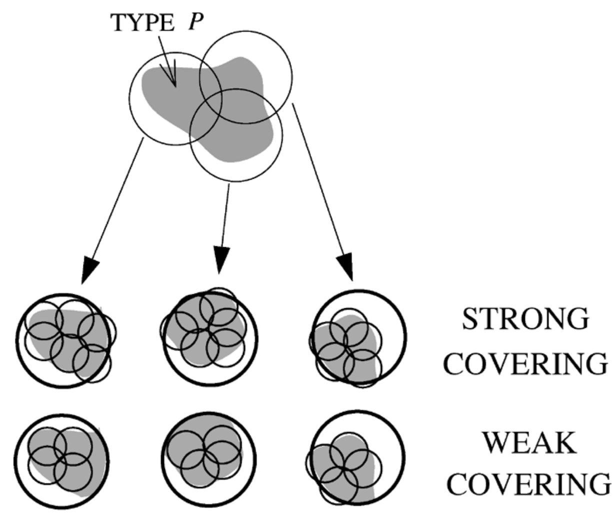

Fig. 1. Comparison of *strong* and *weak* hierarchical type covering schemes. Notice that the relaxed requirement of weak covering, i.e., that every vector $\boldsymbol{x}$ must be jointly covered by a (D_1, D_2)-pair, may result in reduced number of D_2-spheres compared to strong covering.

The concept of *hierarchical* type covering was introduced in [7]. According to the definition therein, in the first stage, type class P is covered using D_1-spheres with respect to (w.r.t.) the distortion measure d_1, and in the second stage, the portion of each D_1-sphere (parent) that lies within type class P is entirely covered using equal number of D_2-spheres (children) w.r.t. d_2. In other words:

i. every vector $\boldsymbol{x}$ in type class P is covered by at least one D_1-sphere, and
ii. among the children of *every* D_1-sphere that covers $\boldsymbol{x}$, there exists at least one D_2-sphere also covering $\boldsymbol{x}$.

In this correspondence, we call this methodology *strong* hierarchical type covering in order to distinguish it from the *weak* version introduced in [15]. In weak covering, equal number of D_2-spheres associated with (i.e., children of) each D_1-sphere are chosen so as to guarantee that

> for every source vector $\boldsymbol{x}$ in type class P, there exists a pair of parent D_1- and child D_2-spheres both covering $\boldsymbol{x}$.

Clearly, this is a more relaxed requirement, and therefore every strong covering also constitutes a weak covering by definition. See Fig. 1 for the illustration of the distinction between strong and weak covering.

The motivation behind the introduction of hierarchical type covering was mainly the determination of achievable error exponents in scalable source coding. In fact, it is easy to verify that weak covering is sufficient for that purpose, because the encoder is in general allowed to search over the entire tree-structured codebook, i.e., over parent-child (D_1, D_2)-pairs that jointly cover the given source vector. Strong hierarchical type covering, on the other hand, is necessary for the generalization of the guessing game, i.e., for hierarchical guessing [12]: In the first stage Alice presents her guesses $\boldsymbol{y}_1(1), \boldsymbol{y}_1(2), \ldots$ until Bob informs her that $d_1(\boldsymbol{x}, \boldsymbol{y}_1(i)) \leq D_1$, and in the second stage, she presents new guesses $\boldsymbol{y}_2(1|i), \boldsymbol{y}_2(2|i), \ldots$, depending on her guesses at the first stage, until $d_2(\boldsymbol{x}, \boldsymbol{y}_2(j|i)) \leq D_2$. Similar to one-stage guessing, the objective is the minimization of $E\{G(\boldsymbol{X})^\rho\}$, where $G(\boldsymbol{X}) = G_1(\boldsymbol{X}) + G_2(\boldsymbol{X})$ is the total number of guesses. The

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motivation behind hierarchical guessing is that good guessing strategies correspond to good search strategies in (i) quantizer codebooks that have a hierarchical structure, and (ii) high-dimensional databases where data is stored in clusters and given a query, the strategy for finding a single D_2-matching vector in the database is to find a cluster representative that D_1-matches the query and to search inside that cluster for a D_2-match.[1] Under the assumption that Alice *knows* the type $P_{\boldsymbol{x}}$ of $\boldsymbol{x}$, the best known strategy (which was claimed to be optimal in [12]) is to find a strong hierarchical covering of $P_{\boldsymbol{x}}$ achieving a certain balance point in the tradeoff of the required number of D_1- and D_2-spheres.

The distinction between strong and weak hierarchical covering motivated us to rederive single-letter characterizations of the rates corresponding to the number of D_1- and D_2-spheres necessary and sufficient to cover a type class in both the weak and the strong senses. The main result of this correspondence is that, somewhat surprisingly, the two characterizations lead to different rate regions, i.e., the rate region of strong covering is contained in that of weak covering (we provide an example where the former is *strictly* contained in the latter). In fact, the claimed rate region for strong covering that appeared in [7], which coincides with the achievability region for scalable source coding [8], [13], turns out to be precisely the achievability region for *weak covering*.[2] Since weak covering is sufficient for the analysis of error exponents, all the results stated in [7] and [15], pertaining to error exponents, remain correct. On the other hand, the result in [12], which effectively assumes the results of [7] as valid for strong covering, needs to be reinvestigated and revised to reflect the true region of achievable rates in strong hierarchical covering.

The achievability results for the two types of covering are best understood in terms of a third, more technical, variant of covering—joint type covering. Given a joint distribution $P_{XY_1Y_2}$ on the source and the reproduction alphabets whose source marginal is P, the goal of joint type covering is to design first and second stage codebooks satisfying the following conditions.

i. For every vector $\boldsymbol{x}$ in type class P, there is some first stage codeword $\boldsymbol{y}_1(i)$ such that the joint type of $(\boldsymbol{x}, \boldsymbol{y}_1(i))$ is P_{XY_1}.
ii. For every pair $(\boldsymbol{x}, \boldsymbol{y}_1(i))$ with joint type P_{XY_1}, there is an element $\boldsymbol{y}_2(j|i)$ of the second stage codebook corresponding to $\boldsymbol{y}_1(i)$ such that the joint type of $(\boldsymbol{x}, \boldsymbol{y}_1(i), \boldsymbol{y}_2(j|i))$ is $P_{XY_1Y_2}$.

Even though this type of covering will be directly employed for proving the forward directions of weak and strong type covering lemmas, it is i) too strong to employ in the derivation of a converse for weak covering, and hence for scalable coding, and ii) too weak to derive a tight converse for strong covering. Joint type covering can also be used in a variant of the hierarchical guessing problem where the guesses need to satisfy joint type requirements instead of distortion requirements (see Section VI).

The organization of the correspondence is as follows. We begin in the next section with the preliminaries and motivation. In this section, we shall also present a clarification of some confusion in the literature regarding the achievable rate region for scalable source coding, leading to a new lower bound on the minimum achievable guessing exponent. We then derive the region of achievable rates for joint type covering in Section III. In Section IV, we present the region of achievable rates for strong hierarchical covering, and demonstrate via an example that it may be strictly contained in the achievable region of scalable source coding. Then we formally show in Section V that the scalable source coding region is indeed the achievable rate region for weak hierarchical covering. Finally, in light of the revised rate region for strong covering, Section VI reinvestigates the upper bound on the minimum achievable guessing exponent.

[1]The latter usage of hierarchical guessing is justified by the fact that due to the curse of dimensionality, efficient indexing schemes are hard to come by and the performance of clustering followed by sequential search over cluster representatives is comparable to the performance of more complicated search strategies [5], [10], [14].

[2]In private communication, Linder, Narayan (coauthor of [7]), and an anonymous reviewer, pointed out the specific error in the proof of the hierarchical type covering lemma in [7]. Basically, the error stems from the assumption that marginally typical $\boldsymbol{x}$ and $\boldsymbol{y}_1$ are also jointly typical when $d_1(\boldsymbol{x}, \boldsymbol{y}_1) \leq D_1$. They also suggested a relaxed variant of strong covering—joint type covering—which we discuss next.

II. Preliminaries and Motivation

A. General Notation

Let $\mathcal{A}$ and $\mathcal{B}$ be finite alphabets. For a given vector $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathcal{A}^n$, the empirical probability mass function, or the *type*, of $\boldsymbol{a}$ is denoted by $P_{\boldsymbol{a}}$, where

$$P_{\boldsymbol{a}}(a) = \frac{1}{n} N(a|\boldsymbol{a}), \quad a \in \mathcal{A}$$

where $N(a|\boldsymbol{a})$ is the number of occurrences of the letter $a \in \mathcal{A}$ in $\boldsymbol{a}$. The set of all vectors $\boldsymbol{a}$ having type $P_{\boldsymbol{a}} = P$ is called the type class P, and is denoted by T_P^n

$$T_P^n = \{\boldsymbol{a} \in \mathcal{A}^n : P_{\boldsymbol{a}} = P\}.$$

Similarly, for given vectors $\boldsymbol{a} \in \mathcal{A}^n$ and $\boldsymbol{b} \in \mathcal{B}^n$, the *conditional type* of $\boldsymbol{b}$ given $\boldsymbol{a}$ is denoted by $V_{\boldsymbol{b}|\boldsymbol{a}}$, where

$$\frac{1}{n} N(a|\boldsymbol{a}) V_{\boldsymbol{b}|\boldsymbol{a}}(b|a) = \frac{1}{n} N(a, b|\boldsymbol{a}, \boldsymbol{b}).$$

The set of vectors $\boldsymbol{b}$ having conditional type $V_{\boldsymbol{b}|\boldsymbol{a}} = V$ is called the V-*shell* of $\boldsymbol{a}$ and will be denoted here by $T_V^n(\boldsymbol{a})$.

Let $\mathcal{M}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B}|\mathcal{A})$ denote the set of all probability distributions on the alphabet $\mathcal{A}$, and the set of all conditional distributions from $\mathcal{A}$ to $\mathcal{B}$, respectively. Also let $\mathcal{M}^n(\mathcal{A})$ denote the set of all valid types of length-n sequences over $\mathcal{A}$, or

$$\mathcal{M}^n(\mathcal{A}) = \{P \in \mathcal{M}(\mathcal{A}) : T_P^n \neq \emptyset\}.$$

Similarly, for $P \in \mathcal{M}^n(\mathcal{A})$, let $\mathcal{C}_P^n(\mathcal{B}|\mathcal{A})$ denote the set of all valid conditional types of vectors $\boldsymbol{b} \in \mathcal{B}^n$ given any[3] $\boldsymbol{a} \in T_P^n$

$$\mathcal{C}_P^n(\mathcal{B}|\mathcal{A}) = \{V \in \mathcal{C}(\mathcal{B}|\mathcal{A}) : T_V^n(\boldsymbol{a}) \neq \emptyset \text{ for } \boldsymbol{a} \in T_P^n\}.$$

An important fact we will use frequently is that $\mathcal{M}^n(\mathcal{A})$ is *dense* in $\mathcal{M}(\mathcal{A})$. That is, for any $P \in \mathcal{M}(\mathcal{A})$ and $\epsilon > 0$, we can find $P_n \in \mathcal{M}^n(\mathcal{A})$ such that $\|P_n - P\| \leq \epsilon$ if $n \geq n(\epsilon)$. Similarly, $\mathcal{C}_P^n(\mathcal{B}|\mathcal{A})$ is dense in $\mathcal{C}(\mathcal{B}|\mathcal{A})$ in the sense that for any $\epsilon > 0, P \in \mathcal{M}^n(\mathcal{A})$, and $V \in \mathcal{C}(\mathcal{B}|\mathcal{A})$, there exists $V_n \in \mathcal{C}_P^n(\mathcal{B}|\mathcal{A})$ such that $\|P \circ V_n - P \circ V\| \leq \epsilon$ if $n \geq n(\epsilon)$.

For $Q \in \mathcal{M}(\mathcal{A})$ and $V \in \mathcal{C}(\mathcal{B}|\mathcal{A})$, we denote by $I(Q, V)$ the mutual information between random variables A and B induced by the joint distribution $Q \circ V$, i.e.,

$$I(Q, V) = \sum_{a \in \mathcal{A}, b \in \mathcal{B}} Q(a) V(b|a) \log \frac{V(b|a)}{\sum_{a'} Q(a') V(b|a')}.$$

[3]The validity of a conditional type V, i.e., whether $T_V^n(\boldsymbol{a}) \neq \emptyset$, depends on $\boldsymbol{a}$ only through its type $P_{\boldsymbol{a}}$.

Here and in the sequel, all logarithms are natural. Also, let $H(Q)$ and $H(V|Q)$ respectively denote the entropy of A, and the conditional entropy of B given A, i.e.,

$$H(Q) = -\sum_{a\in\mathcal{A}} Q(a)\log Q(a)$$

and

$$H(V|Q) = -\sum_{a\in\mathcal{A},b\in\mathcal{B}} Q(a)V(b|a)\log V(b|a).$$

For $Q_1 \in \mathcal{M}(\mathcal{A}_1), Q_{2|1} \in \mathcal{C}(\mathcal{A}_2|\mathcal{A}_1)$, and $W \in \mathcal{C}(\mathcal{B}|\mathcal{A}_1 \times \mathcal{A}_2)$, we denote by $I(Q_{2|1}, W|Q_1)$ the mutual information between random variables A_2 and B given A_1, induced by the joint distribution $Q_1 \circ Q_{2|1} \circ W$, i.e.,

$$I(Q_{2|1}, W|Q_1) = \sum_{a_1\in\mathcal{A}_1,a_2\in\mathcal{A}_2,b\in\mathcal{B}} Q_1(a_1)Q_{2|1}(a_2|a_1)W(b|a_1,a_2) \cdot \log\frac{W(b|a_1,a_2)}{\sum_{a_2'} W(b|a_1,a_2')Q_{2|1}(a_2'|a_1)}.$$

Note that all these entropy and mutual information functionals are finite-valued and *continuous* in their parameters.

B. Basic Properties of Types

We will repeat here for convenience some very basic properties of types. For a more detailed discussion, see [4].

A fundamental property of types is given by the type counting lemma [4, Lemma 1.2.2], which states that there are only polynomially many elements in the set $\mathcal{M}^n(\mathcal{A})$. More precisely,

$$|\mathcal{M}^n(\mathcal{A})| \le (n+1)^{|\mathcal{A}|}.$$

Similarly, for $Q \in \mathcal{M}^n(\mathcal{A})$

$$|\mathcal{C}_Q^n(\mathcal{B}|\mathcal{A})| \le (n+1)^{|\mathcal{A}||\mathcal{B}|}.$$

Another useful property for our purposes concerns the size of type classes [4 Lemmas 1.2.3 and 1.2.5]: For any $Q \in \mathcal{M}^n(\mathcal{A})$, we have

$$H(Q) - |\mathcal{A}|\delta(n) \le \frac{1}{n}\log|T_Q^n| \le H(Q) \quad (1)$$

where

$$\delta(n) \triangleq \frac{1}{n}\log(n+1).$$

Similarly, for $\boldsymbol{a} \in \mathcal{A}^n$ and $V \in \mathcal{C}_{P_{\boldsymbol{a}}}^n(\mathcal{B}|\mathcal{A})$, we have

$$H(V|P_{\boldsymbol{a}}) - |\mathcal{A}||\mathcal{B}|\delta(n) \le \frac{1}{n}\log|T_V^n(\boldsymbol{a})| \le H(V|P_{\boldsymbol{a}}). \quad (2)$$

Observe that $\delta(n) \longrightarrow 0$ as $n \longrightarrow \infty$, and therefore the above estimates of the sizes of type classes and V-shells become increasingly accurate as n grows large.

C. Lossy Source Coding

Let $\{X_t\}_{t=1}^{\infty}$ be a DMS with alphabet $\mathcal{X}$, i.e., the samples $X_t \in \mathcal{X}$ are independent and identically distributed (i.i.d.) with probability mass function (pmf) $P \in \mathcal{M}(\mathcal{X})$. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be two reproduction alphabets. Define the distortion between sequences $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y}_k = (y_{k1}, \dots, y_{kn})$ for $k = 1, 2$ as

$$d_k(\boldsymbol{x}, \boldsymbol{y}_k) = \frac{1}{n}\sum_{t=1}^{n} d_k(x_t, y_{kt})$$

where $d_k : \mathcal{X} \times \mathcal{Y}_k \to [0,\infty)$ are distortion measures satisfying $\min_{y_k} d_k(x, y_k) = 0$ for all $x \in \mathcal{X}$. Denote by $\mathcal{S}_k(\boldsymbol{y}_k, D_k)$ the "sphere" of radius D_k about $\boldsymbol{y}_k \in \mathcal{Y}_k$ for $k = 1, 2$, i.e.,

$$\mathcal{S}_k(\boldsymbol{y}_k, D_k) = \{\boldsymbol{x} \in \mathcal{X}^n : d_k(\boldsymbol{x}, \boldsymbol{y}_k) \le D_k\}.$$

We say that $\boldsymbol{y}_k$ "D_k-matches" with $\boldsymbol{x}$ if

$$\boldsymbol{x} \in \mathcal{S}_k(\boldsymbol{y}_k, D_k).$$

A length-n block code consists of an encoder-decoder pair

$$f_1: \mathcal{X}^n \to \{1, \dots, M_1\}$$

and

$$g_1: \{1, \dots, M_1\} \to \mathcal{Y}_1^n.$$

The pair (R_1, D_1) is said to be *achievable* if for every $\epsilon > 0$, there exists a block code with large enough n such that

$$\frac{1}{n}\log M_1 \le R_1 + \epsilon$$
$$\Pr[d_1(\boldsymbol{X}, g_1(f_1(\boldsymbol{X}))) \le D_1] \ge 1 - \epsilon.$$

For a reproduction pmf $Q_1 \in \mathcal{M}(\mathcal{Y}_1)$, we denote by $\mathcal{V}(P, Q_1, D_1)$ the set of "backward channel" pmfs $V \in \mathcal{C}(\mathcal{X}|\mathcal{Y}_1)$ which are consistent with the source pmf P, and at the same time yield an expected distortion of at most D_1 between the induced random variables X and Y_1 :

$$\mathcal{V}(P, Q_1, D_1) \triangleq \Bigg\{V(x|y_1) : \sum_{y_1\in\mathcal{Y}_1} Q_1(y_1)V(x|y_1) = P(x), \sum_{x\in\mathcal{X},y_1\in\mathcal{Y}_1} Q_1(y_1)V(x|y_1)d_1(x,y_1) \le D_1\Bigg\}.$$

It is well-known that a pair (R_1, D_1) is achievable if and only if

$$I(Q_1, V) \le R_1 \quad (3)$$

for some $Q_1 \in \mathcal{M}(\mathcal{Y}_1)$ and $V \in \mathcal{V}(P, Q_1, D_1)$.

A two-stage code is obtained by adding a refinement encoder-decoder pair on top of the single-stage pair (f_1, g_1)

$$f_2 : \mathcal{X}^n \to \{1, \dots, M_2\}$$

and

$$g_2 : \{1, \dots, M_1\} \times \{1, \dots, M_2\} \to \mathcal{Y}_2^n.$$

The quadruple (R_1, R_2, D_1, D_2) is said to be achievable if for every $\epsilon > 0$, there exists a two-stage block code with large enough n such that

$$\frac{1}{n}\log M_1 \le R_1 + \epsilon$$
$$\frac{1}{n}\log M_2 \le R_2 + \epsilon$$

and

$$\Pr[d_1(\boldsymbol{X}, g_1(f_1(\boldsymbol{X}))) \le D_1,\; d_2(\boldsymbol{X}, g_2(f_1(\boldsymbol{X}), f_2(\boldsymbol{X}))) \le D_2] \ge 1 - \epsilon.$$

For a first stage reproduction pmf $Q_1 \in \mathcal{M}(\mathcal{Y}_1)$, a refinement pmf $Q_{2|1} \in \mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)$, and a backward channel pmf $V \in \mathcal{C}(\mathcal{X}|\mathcal{Y}_1)$, define

$$\mathcal{W}(V, Q_{2|1}, Q_1, D_2) \triangleq \Bigg\{ W(x|y_1, y_2) : \sum_{y_2 \in \mathcal{Y}_2} Q_{2|1}(y_2|y_1) W(x|y_1, y_2) = V(x|y_1), \sum_{x\in\mathcal{X}, y_1\in\mathcal{Y}_1, y_2\in\mathcal{Y}_2} \cdot\, Q_1(y_1) Q_{2|1}(y_2|y_1) W(x|y_1,y_2) d_2(x, y_2) \le D_2 \Bigg\}$$

i.e., the set of all backward channel pmfs $W \in \mathcal{C}(\mathcal{X}|\mathcal{Y}_1 \times \mathcal{Y}_2)$ which are consistent with V and yield an expected distortion of at most D_2 between the induced random variables X and Y_2. It was shown in [8], [13] that a quadruple (R_1, R_2, D_1, D_2) is achievable if and only if

$$I(Q_1, V) \le R_1$$
$$I(Q_1, V) + I(Q_{2|1}, W|Q_1) \le R_1 + R_2 \tag{4}$$

for some $Q_1 \in \mathcal{M}(\mathcal{Y}_1), Q_{2|1} \in \mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1), V \in \mathcal{V}(P, Q_1, D_1)$, and $W \in \mathcal{W}(V, Q_{2|1}, Q_1, D_2)$. We will denote by $\mathcal{R}_P(D_1, D_2)$ the region of all achievable rates (R_1, R_2) in two-stage coding of source P with prescribed distortion values (D_1, D_2).

At this point, we would like to make explicit a distinction that has hitherto led to some confusion in the literature. Given distortion values (D_1, D_2), we denote by $\mathcal{R}'_P(D_1, D_2)$ the region of all rates (R_1, R_2) satisfying

$$I(Q_1, V) \le R_1$$
$$I(Q_{2|1}, W|Q_1) \le R_2. \tag{5}$$

Note that if a rate pair (R_1, R_2) is in $\mathcal{R}'_P(D_1, D_2)$, it is also in $\mathcal{R}_P(D_1, D_2)$, but the converse is not true. Were $\mathcal{R}'_P(D_1, D_2)$ the true "converse" region for the scalable coding problem, achievability of any (R_1, R_2) would imply the existence of $Q_1 \in \mathcal{M}(\mathcal{Y}_1)$, $Q_{2|1} \in \mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1), V \in \mathcal{V}(P, Q_1, D_1)$, and $W \in \mathcal{W}(V, Q_{2|1}, Q_1, D_2)$, satisfying

$$I(Q_1, V) \le R_1 + R_2$$
$$I(Q_{2|1}, W|Q_1) = 0$$

as one can always transfer rate from the second stage to the first stage while maintaining a constant total rate. We construct an example in Appendix A where $I(Q_{2|1}, W|Q_1) = 0$ is impossible to satisfy, thereby proving $\mathcal{R}'_P(D_1, D_2) \ne \mathcal{R}_P(D_1, D_2)$ in general.

D. Matching Probabilities for Pure-Type Codebooks

Consider a two-stage coding scenario, where codevectors $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ are randomly generated from "pure types" $T^n_{Q_1}$ at the first stage, and $T^n_{Q_{2|1}}(\boldsymbol{Y}_1)$ in the second stage, respectively. Then the probabilities

$$\Pr[d_1(\boldsymbol{x}, \boldsymbol{Y}_1) \le D_1]$$

and

$$\Pr[d_2(\boldsymbol{x}, \boldsymbol{Y}_2) \le D_2 | \boldsymbol{Y}_1 = \boldsymbol{y}_1]$$

decay exponentially fast in n for any $\boldsymbol{x}$. The exponent of the D_1-matching probability $\Pr[d_1(\boldsymbol{x}, \boldsymbol{Y}_1) \le D_1]$ is well-known [17], [18]. Here, we will rederive that exponent and present a straightforward extension for the exponent of $\Pr[d_2(\boldsymbol{x}, \boldsymbol{Y}_2) \le D_2 | \boldsymbol{Y}_1 = \boldsymbol{y}_1]$.

Following the notation of [18], we denote by $I_m(P\|Q_1, D_1)$ the *lower mutual information*

$$I_m(P\|Q_1, D_1) \triangleq \inf_{V \in \mathcal{V}(P, Q_1, D_1)} I(Q_1, V). \tag{6}$$

If the set $\mathcal{V}(P, Q_1, D_1)$ is empty, then it is understood that $I_m(P\|Q_1, D_1) = \infty$. Otherwise, the infimum of (6) can be replaced by a minimum since $I(Q_1, V)$ is finite and continuous in V, and the set $\mathcal{V}(P, Q_1, D_1)$ is bounded and closed. We will denote by $V^*(P, Q_1, D_1)$ the backward channel V minimizing (6).

We similarly define the *conditional* lower mutual information $I_m(V\|Q_{2|1}, Q_1, D_2)$

$$I_m(V\|Q_{2|1}, Q_1, D_2) \triangleq \inf_{W \in \mathcal{W}(V, Q_{2|1}, Q_1, D_2)} I(Q_{2|1}, W|Q_1). \tag{7}$$

Similarly, if $\mathcal{W}(V, Q_{2|1}, Q_1, D_2)$ is empty, then it is understood that $I_m(V\|Q_{2|1}, Q_1, D_2) = \infty$, and otherwise the infimum in (7) can be replaced by a minimum. Denote by $W^*(V, Q_{2|1}, Q_1, D_2)$ the minimum W achieving (7).

Let us also define the counterparts of the sets $\mathcal{V}(P, Q_1, D_1)$ and $\mathcal{W}(V, Q_{2|1}, Q_1, D_2)$ in the domain of types

$$\mathcal{V}^n(P, Q_1, D_1) \triangleq \mathcal{V}(P, Q_1, D_1) \cap \mathcal{C}^n_{Q_1}(\mathcal{X}|\mathcal{Y}_1) \tag{8}$$

$$\mathcal{W}^n(V, Q_{2|1}, Q_1, D_2) \triangleq \mathcal{W}(V, Q_{2|1}, Q_1, D_2) \times \cap \mathcal{C}^n_{Q_1 \circ Q_{2|1}}(\mathcal{X}|\mathcal{Y}_1 \times \mathcal{Y}_2) \tag{9}$$

where (9) is defined only when $V \in \mathcal{C}^n_{Q_1}(\mathcal{X}|\mathcal{Y}_1)$.

Let $P \in \mathcal{M}^n(\mathcal{X})$ and $Q_1 \in \mathcal{M}^n(\mathcal{Y}_1)$. Suppose, for any source vector $\boldsymbol{x} \in T^n_P$, we randomly pick a codevector $\boldsymbol{Y}_1$, with a uniform distribution over $T^n_{Q_1}$. That is,

$$\Pr[\boldsymbol{Y}_1 = \boldsymbol{y}_1] = \frac{\mathbf{1}(\boldsymbol{y}_1 \in T^n_{Q_1})}{|T^n_{Q_1}|}.$$

Then for any conditional type $V \in \mathcal{C}^n_{Q_1}(\mathcal{X}|\mathcal{Y}_1)$ satisfying $\sum_{y_1} Q_1(y_1)V(x|y_1) = P(x)$, we can write

$$e^{-n[I(Q_1,V)+|\mathcal{X}||\mathcal{Y}_1|\delta(n)]} \le \Pr[\boldsymbol{x} \in T^n_V(\boldsymbol{Y}_1)] \le e^{-n[I(Q_1,V)-|\mathcal{Y}_1|\delta(n)]} \tag{10}$$

using (1) and (2). The total probability of a D_1-match of the codeword $\boldsymbol{Y}_1$ with $\boldsymbol{x}$ is given by

$$\Pr[d_1(\boldsymbol{x}, \boldsymbol{Y}_1) \le D_1] = \sum_{V \in \mathcal{V}^n(P, Q_1, D_1)} \Pr[\boldsymbol{x} \in T^n_V(\boldsymbol{Y}_1)]. \tag{11}$$

Consider the type $V^*_n \in \mathcal{V}^n(P, Q_1, D_1)$ minimizing $I(Q_1, V)$. Since the set $\mathcal{C}^n_{Q_1}(\mathcal{X}|\mathcal{Y}_1)$ is dense in $\mathcal{C}(\mathcal{X}|\mathcal{Y}_1)$, $Q_1 \circ V^*_n$ approaches

$Q_1 \circ V^*(P, Q_1, D_1)$ for large n. It then follows from the continuity of $I(Q_1, V)$ that

$$I_m(P\|Q_1, D_1) \le I(Q_1, V_n^*) \le I_m(P\|Q_1, D_1) + \epsilon$$

for arbitrary $\epsilon > 0$ and $n \ge n_0(\epsilon)$. Thus, we can write

$$\begin{aligned} e^{-n[I_m(P\|Q_1,D_1)+2\epsilon]} &\le \Pr[d_1(\boldsymbol{x}, \boldsymbol{Y}_1) \le D_1] \\ &\le e^{-n[I_m(P\|Q_1,D_1)-2\epsilon]} \end{aligned} \tag{12}$$

for all $n \ge n_1(\epsilon)$. The lower bound in (12) follows by dropping all types other than V_n^* from the summation in (11). To prove the upper bound, we can use the fact that there are at most polynomially many conditional types in $\mathcal{C}_{Q_1}^n(\mathcal{X}|\mathcal{Y}_1)$, and in effect, the summation in (11) can be replaced by a maximum operation.

Consider now the extension of the above scheme to two-stage source coding. Suppose that for a fixed first stage codevector $\boldsymbol{y}_1 \in T_{Q_1}^n$, we pick the second stage codevector $\boldsymbol{Y}_2$ with a uniform distribution over $T_{Q_{2|1}}^n(\boldsymbol{y}_1)$, where $Q_{2|1} \in \mathcal{C}_{Q_1}^n(\mathcal{Y}_2|\mathcal{Y}_1)$

$$\Pr[\boldsymbol{Y}_2 = \boldsymbol{y}_2] = \frac{\mathbf{1}\left(\boldsymbol{y}_2 \in T_{Q_{2|1}}^n(\boldsymbol{y}_1)\right)}{\left|T_{Q_{2|1}}^n(\boldsymbol{y}_1)\right|}.$$

Now, for any source vector $\boldsymbol{x} \in T_V^n(\boldsymbol{y}_1)$, and any conditional type $W \in \mathcal{C}_{Q_1 \circ Q_{2|1}}^n(\mathcal{X}|\mathcal{Y}_1 \times \mathcal{Y}_2)$ satisfying

$$\sum_{y_2} Q_{2|1}(y_2|y_1) W(x|y_1, y_2) = V(x|y_1)$$

we have

$$e^{-n[I(Q_{2|1}, W|Q_1)+|\mathcal{X}||\mathcal{Y}_1||\mathcal{Y}_2|\delta(n)]} \le \Pr[\boldsymbol{x} \in T_W^n(\boldsymbol{y}_1, \boldsymbol{Y}_2)] \tag{13}$$

and

$$\Pr[\boldsymbol{x} \in T_W^n(\boldsymbol{y}_1, \boldsymbol{Y}_2)] \le e^{-n[I(Q_{2|1}, W|Q_1)-|\mathcal{Y}_1||\mathcal{Y}_2|\delta(n)]}. \tag{14}$$

Using the relation

$$\begin{aligned} \Pr[d_2(\boldsymbol{x}, \boldsymbol{Y}_2) &\le D_2 | \boldsymbol{Y}_1 = \boldsymbol{y}_1] \\ &= \sum_{W \in \mathcal{W}^n(V, Q_{2|1}, Q_1, D_2)} \Pr\left[\boldsymbol{x} \in T_W^n(\boldsymbol{y}_1, \boldsymbol{Y}_2)\right] \end{aligned}$$

and with similar analysis as above, we conclude that

$$e^{-n[I_m(V\|Q_{2|1}, Q_1, D_2)+2\epsilon]} \le \Pr[d_2(\boldsymbol{x}, \boldsymbol{Y}_2) \le D_2 | \boldsymbol{Y}_1 = \boldsymbol{y}_1] \tag{15}$$

and

$$\Pr[d_2(\boldsymbol{x}, \boldsymbol{Y}_2) \le D_2 | \boldsymbol{Y}_1 = \boldsymbol{y}_1] \le e^{-n[I_m(V\|Q_{2|1}, Q_1, D_2)-2\epsilon]} \tag{16}$$

for all $\epsilon > 0$ and $n \ge n_2(\epsilon)$.

E. Guessing Subject to Distortion

For the DMS $\{X_t\}_{t=1}^\infty$ with pmf P, let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ this time denote *guessing* alphabets. A (fixed) list $\mathcal{G} = \{\boldsymbol{y}_1(1), \boldsymbol{y}_1(2), \ldots\}$ is called a D_1-admissible guessing strategy if

$$\bigcup_i \mathcal{S}_1(\boldsymbol{y}_1(i), D_1) = \mathcal{X}^n.$$

The guessing function $G(\boldsymbol{x})$ induced by a D_1-admissible strategy is the function that maps each $\boldsymbol{x} \in \mathcal{X}^n$ into a positive integer indicating the first guessing codevector $\boldsymbol{y}_1(i) \in \mathcal{G}$ that D_1-matches $\boldsymbol{x}$.

Given an intermediate distortion level D_1, and a target distortion level D_2, a two-stage (D_1, D_2)-admissible guessing strategy comprises a D_1-admissible guessing strategy $\mathcal{G}_1 = \{\boldsymbol{y}_1(1), \boldsymbol{y}_1(2), \ldots\}$ with a guessing function $G_1(\boldsymbol{x})$, and a set of lists $\mathcal{G}_2(i) = \{\boldsymbol{y}_2(1|i), \boldsymbol{y}_2(2|i), \ldots\}$ for $i = 1, 2, \ldots$, such that for each i,

$$\bigcup_j \mathcal{S}_2(\boldsymbol{y}_2(j|i), D_2) \supseteq \mathcal{S}_1(\boldsymbol{y}_1(i), D_1) \cap \left[\bigcup_{l=1}^{i-1} \mathcal{S}_1(\boldsymbol{y}_1(l), D_1)\right]^c.$$

Denoting by $G_2(\boldsymbol{x})$ the index j of the first codevector $\boldsymbol{y}_2(j|G_1(\boldsymbol{x}))$ that D_2-matches $\boldsymbol{x}$, the guessing function induced by a (D_1, D_2)-admissible strategy is $G(\boldsymbol{x}) = G_1(\boldsymbol{x}) + G_2(\boldsymbol{x})$.

The main problem in guessing subject to distortion is the determination of the optimum ρth-order guessing exponent

$$\mathcal{E}(D_1, D_2, \rho) = \lim_{n \to \infty} \frac{1}{n} \min_{\mathcal{G}_1, \mathcal{G}_2} \log E\{G(\boldsymbol{X})^\rho\} \tag{17}$$

whenever the limit exists.

In [12], using the scalable coding converse, a converse result for $\mathcal{E}(D_1, D_2, \rho)$ was presented

$$\begin{aligned} \liminf_{n \to \infty} \frac{1}{n} &\min_{\mathcal{G}_1, \mathcal{G}_2} \log E\{G(\boldsymbol{X})^\rho\} \\ &\ge \max_{P' \in \mathcal{M}(\mathcal{X})} [\rho K(D_1, D_2, P') - \mathcal{D}(P'\|P)] \end{aligned} \tag{18}$$

where

$$\begin{aligned} &K(D_1, D_2, P) \\ &\triangleq \min_{\substack{Q_1 \in \mathcal{M}(\mathcal{Y}_1) \\ Q_{2|1} \in \mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1) \\ V \in \mathcal{V}(P, Q_1, D_1) \\ W \in \mathcal{W}(V, Q_{2|1}, Q_1, D_2)}} \max\{I(Q_1, V), I(Q_{2|1}, W|Q_1)\} \end{aligned} \tag{19}$$

and $\mathcal{D}(P'\|P)$ denotes the standard Kullback-Leibler divergence

$$\mathcal{D}(P'\|P) = \sum_{x \in \mathcal{X}} P'(x) \log \frac{P'(x)}{P(x)}.$$

Relying on the hierarchical type covering lemma of [7], the authors of [12] also proved that the right-hand side (RHS) of (18) is achievable by a guessing strategy if $P_{\boldsymbol{x}}$, the type of $\boldsymbol{x}$, is known beforehand. To that end, they used a strong hierarchical covering of type $P_{\boldsymbol{x}}$. This proof, together with (18), suggested that the hierarchical guessing exponent is completely characterized (at least, for the case where $P_{\boldsymbol{x}}$ is known to the guesser.) However, in light of our observations and results herein, two main difficulties arise. First, the proof of (18), which appeared in [12], assumes that the scalable source coding converse region is $\mathcal{R}'_P(D_1, D_2)$. But, as we have shown earlier, the true converse region is $\mathcal{R}_P(D_1, D_2)$, which is generally distinct from $\mathcal{R}'_P(D_1, D_2)$. This observation leads to the revised scalable coding converse[4]

$$\begin{aligned} \liminf_{n \to \infty} \frac{1}{n} &\min_{\mathcal{G}_1, \mathcal{G}_2} \log E\{G(\boldsymbol{X})^\rho\} \\ &\ge \max_{P' \in \mathcal{M}(\mathcal{X})} [\rho J(D_1, D_2, P') - \mathcal{D}(P'\|P)] \end{aligned} \tag{20}$$

with

$$\begin{aligned} &J(D_1, D_2, P) \\ &\triangleq \min \max \left\{ I(Q_1, V), \frac{I(Q_1, V) + I(Q_{2|1}, W|Q_1)}{2} \right\} \end{aligned} \tag{21}$$

where the minimization is over the same set as in (19). Note that $K(D_1, D_2, P) \ge J(D_1, D_2, P)$, which implies that the RHS of (20) is greater than or equal to that of (18). We further show in Appendix B that the inequality can be strict.

[4]The best lower bound to $\max\{a, b\}$ when $a \ge c$ and $a + b \ge c + d$ is given by $\max\{c, \frac{c+d}{2}\}$. We use this fact with $a = R_1, b = R_2, c = I(Q_1, V)$, and $d = I(Q_{2|1}, W|Q_1)$.

Moreover, since [12] relies on the hierarchical type covering lemma of [7], the achievability result must also be reinvestigated. After characterizing the correct achievable strong covering rates, we will restate the best achievable guessing exponent based on strong covering of $P_{\boldsymbol{x}}$ to be

$$\max_{P' \in \mathcal{M}(\mathcal{X})} [\rho L(D_1, D_2, P') - \mathcal{D}(P' \| P)] \tag{22}$$

with $L(D_1, D_2, P) \geq K(D_1, D_2, P)$, where $L(D_1, D_2, P)$ will be defined in Section VI. In the same section, we will also demonstrate an example satisfying $L(D_1, D_2, P) > K(D_1, D_2, P)$.

F. Two-Stage Type Covering

We now formally define the two-stage type covering methodologies we will compare.

Definition 1: A type $P \in \mathcal{M}^n(\mathcal{X})$ is said to be *weakly* (D_1, D_2)-*covered* by codebooks $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1}$ and $\{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2}$ for $1 \leq i \leq M_1$, if for all $\boldsymbol{x} \in T_P^n$, there exists a pair $(i, j) \in \{1, \ldots, M_1\} \times \{1, \ldots, M_2\}$ with

$$d_1(\boldsymbol{x}, \boldsymbol{y}_1(i)) \leq D_1$$
$$d_2(\boldsymbol{x}, \boldsymbol{y}_2(j|i)) \leq D_2.$$

This two-stage type covering strategy is especially useful for lossy source coding purposes. That is, if the codebook pair $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1}$ and $\{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2}$ weakly (D_1, D_2)-covers type P, then it is possible to construct a two-stage coder (f_1, f_2, g_1, g_2) such that

$$d_1(\boldsymbol{x}, g_1(f_1(\boldsymbol{x}))) \leq D_1$$
$$d_2(\boldsymbol{x}, g_2(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))) \leq D_2$$

for all $\boldsymbol{x} \in T_P^n$, the expended rates at the two stages respectively being $\frac{1}{n}\log M_1$ and $\frac{1}{n}\log M_2$.

Definition 2: A type $P \in \mathcal{M}^n(\mathcal{X})$ is said to be *strongly* (D_1, D_2)-*covered* by codebooks $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1}$ and $\{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2}$ for $1 \leq i \leq M_1$, if for all $\boldsymbol{x} \in T_P^n$, there exists $i \in \{1, \ldots, M_1\}$ with

$$d_1(\boldsymbol{x}, \boldsymbol{y}_1(i)) \leq D_1$$

and for all i such that $d_1(\boldsymbol{x}, \boldsymbol{y}_1(i)) \leq D_1$, there exists $j \in \{1, \ldots, M_2\}$ with

$$d_2(\boldsymbol{x}, \boldsymbol{y}_2(j|i)) \leq D_2.$$

In other words, type P is strongly (D_1, D_2)-covered if

$$T_P^n \subseteq \bigcup_{i=1}^{M_1} \mathcal{S}_1(\boldsymbol{y}_1(i), D_1)$$

and

$$T_P^n \cap \mathcal{S}_1(\boldsymbol{y}_1(i), D_1) \subseteq \bigcup_{j=1}^{M_2} \mathcal{S}_2(\boldsymbol{y}_2(j|i), D_2)$$

for all $i \in \{1, \ldots, M_1\}$.

Strong (D_1, D_2)-covering is useful for applications where weak covering is not sufficient, e.g., in hierarchical guessing, as discussed in [12]. Of course, a strong (D_1, D_2)-covering is automatically a weak (D_1, D_2)-covering as well. However, strong covering is unnecessary for source coding purposes, and in fact, as we show in this correspondence, the expended rates for strong covering can be higher than those for weak covering.

We will also investigate the performance of joint type covering of T_P^n.

Definition 3: Let the joint type $P_{XY_1Y_2} \in \mathcal{M}^n(\mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2)$ induce $P \in \mathcal{M}^n(\mathcal{X}), Q_1 \in \mathcal{M}^n(\mathcal{Y}_1), Q_{2|1} \in \mathcal{C}_{Q_1}^n(\mathcal{Y}_2|\mathcal{Y}_1), V \in \mathcal{C}_{Q_1}^n(\mathcal{X}|\mathcal{Y}_1)$, and $W \in \mathcal{C}_{Q_1 \circ Q_{2|1}}^n(\mathcal{X}|\mathcal{Y}_1 \times \mathcal{Y}_2)$ as its corresponding marginal and conditional types. Then P is said to be $P_{XY_1Y_2}$-*covered* by codebooks $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1} \subset T_{Q_1}^n$ and $\{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2} \subset T_{Q_{2|1}}^n(\boldsymbol{y}_1(i))$ for $1 \leq i \leq M_1$, if

$$T_P^n \subseteq \bigcup_{i=1}^{M_1} T_V^n(\boldsymbol{y}_1(i))$$

and

$$T_V^n(\boldsymbol{y}_1(i)) \subseteq \bigcup_{j=1}^{M_2} T_W^n(\boldsymbol{y}_1(i), \boldsymbol{y}_2(j|i))$$

for all $i \in \{1, \ldots, M_1\}$.

Note that the codebook structure is akin to that of strong covering in that for *every* first stage codeword $\boldsymbol{y}_1(i)$ that V-matches $\boldsymbol{x}$, there is at least one second stage codeword $\boldsymbol{y}_2(j|i)$ that along with $\boldsymbol{y}_1(i)$, W-matches $\boldsymbol{x}$.

By judiciously choosing the joint distribution $P_{XY_1Y_2}$, this type of covering can be used to create a stronger scalable source coder as well as a weaker variant of hierarchical guessing where Bob is allowed to inform Alice whether $\boldsymbol{x} \in T_V^n(\boldsymbol{y}_1(i))$ in the first stage and $\boldsymbol{x} \in T_W^n(\boldsymbol{y}_1(i), \boldsymbol{y}_2(j|i))$ in the second stage. However, it cannot be directly used in the original hierarchical guessing introduced in [12]. Also, in general, it results in higher rates in scalable coding, as we will show in Section III.

III. Achievable Rates in Joint Type Covering

The following lemma presents an achievable rate region for joint typicality covering.

Lemma 1: For any joint type $P_{XY_1Y_2} \in \mathcal{M}^n(\mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2)$ inducing $P, Q_1, Q_{2|1}, V$, and W, there exist codebooks $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1} \subset T_{Q_1}^n$ and $\{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2} \subset T_{Q_{2|1}}^n(\boldsymbol{y}_1(i))$ with

$$\frac{1}{n}\log M_1 \leq I(Q_1, V) + \epsilon \tag{23}$$
$$\frac{1}{n}\log M_2 \leq I(Q_{2|1}, W|Q_1) + \epsilon \tag{24}$$

$P_{XY_1Y_2}$-covering T_P^n, for arbitrary $\epsilon > 0$ and large enough n.

Proof: We will first show for large n the existence of a codebook $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1} \subset T_{Q_1}^n$ such that

$$T_P^n \subseteq \bigcup_{i=1}^{M_1} T_V^n(\boldsymbol{y}_1(i))$$

with

$$\frac{1}{n}\log M_1 \leq I(Q_1, V) + \epsilon. \tag{25}$$

Then, for each $i \in \{1, \ldots, M_1\}$, we will show for large n the existence of V-shell refinement codebooks $\{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2} \subset T_{Q_{2|1}}^n(\boldsymbol{y}_1(i))$ such that

$$T_V^n(\boldsymbol{y}_1(i)) \subseteq \bigcup_{j=1}^{M_2} T_W^n(\boldsymbol{y}_1(i), \boldsymbol{y}_2(j|i))$$

with

$$\frac{1}{n}\log M_2 \leq I(Q_{2|1}, W|Q_1) + \epsilon.$$

For any set $\mathcal{B}_1 = \{\boldsymbol{y}_1(i)\}_{i=1}^{M_1}$, denote by $\mathcal{U}(\mathcal{B}_1)$ the set of typical vectors $\boldsymbol{x} \in T_P^n$ which are not "covered" by any $T_V^n(\boldsymbol{y}_1(i))$, i.e.,

$$\mathcal{U}(\mathcal{B}_1) = T_P^n - \bigcup_{i=1}^{M_1} T_V^n(\boldsymbol{y}_1(i)).$$

Similarly, for a V-shell refinement codebook $\mathcal{B}_2(i) = \{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2}$, let $\mathcal{U}(\mathcal{B}_2(i))$ denote the vectors $\boldsymbol{x} \in T_V^n(\boldsymbol{y}_1(i))$ which are not covered by any $T_W^n(\boldsymbol{y}_1(i), \boldsymbol{y}_2(j|i))$, i.e.,

$$\mathcal{U}(\mathcal{B}_2(i)) = T_V^n(\boldsymbol{y}_1(i)) - \bigcup_{j=1}^{M_2} T_W^n(\boldsymbol{y}_1(i), \boldsymbol{y}_2(j|i)).$$

We need to demonstrate the existence of $\mathcal{B}_1$ and $\mathcal{B}_2$ such that

$$|\mathcal{U}(\mathcal{B}_1)| = 0 \tag{26}$$

and

$$|\mathcal{U}(\mathcal{B}_2(i))| = 0, \qquad \forall i \in \{1, \ldots, M_1\}. \tag{27}$$

We follow standard random coding arguments: Randomly pick elements for $\mathcal{B}_1 = \{\boldsymbol{Y}_1(i)\}_{i=1}^{M_1}$ independently and uniformly from $T_{Q_1}^n$. It suffices to show that this random choice of $\mathcal{B}_1$ satisfies $E\{|\mathcal{U}(\mathcal{B}_1)|\} < 1$ to prove the *existence* of a first-stage codebook $\mathcal{B}_1 = \{\boldsymbol{y}_1(i)\}_{i=1}^{M_1} \subset T_{Q_1}^n$ such that (26) holds.

Then, for each $\boldsymbol{y}_1(i)$ in that "good" $\mathcal{B}_1$, randomly pick elements for $\mathcal{B}_2(i) = \{\boldsymbol{Y}_2(j|i)\}_{j=1}^{M_2}$ independently and uniformly from $T_{Q_{2|1}}^n(\boldsymbol{y}_1(i))$. Similarly, it suffices to show that this random choice of $\mathcal{B}_2(i)$ satisfies $E\{|\mathcal{U}(\mathcal{B}_2(i))|\} < 1$ for all $i \in \{1, \ldots, M_1\}$, to prove the existence of some $\mathcal{B}_2$ such that (27) holds. Now

$$\begin{aligned} E\{|\mathcal{U}(\mathcal{B}_1)|\} &= \sum_{\boldsymbol{x} \in T_P^n} \Pr[\boldsymbol{x} \in \mathcal{U}(\mathcal{B}_1)] \\ &= \sum_{\boldsymbol{x} \in T_P^n} (1 - \Pr[\boldsymbol{x} \in T_V^n(\boldsymbol{Y}_1(1))])^{M_1} \\ &\le |T_P^n| \left(1 - e^{-n[I(Q_1,V)+\epsilon/2]}\right)^{M_1} \end{aligned}$$

where in the last inequality, we used (10). Using the identity $(1-t)^K \le e^{-tK}$ for $0 \le t \le 1$, we obtain

$$E\{|\mathcal{U}(\mathcal{B}_1)|\} \le e^{nH(P) - M_1 e^{-n[I(Q_1,V)+\epsilon/2]}}.$$

Hence, choosing

$$I(Q_1, V) + \frac{3\epsilon}{4} \le \frac{1}{n} \log M_1 \le I(Q_1, V) + \epsilon$$

results in $E\{|\mathcal{U}(\mathcal{B}_1)|\} < 1$ for large enough n.

Similarly, for any $i \in \{1, \ldots, M_1\}$

$$\begin{aligned} &E\{|\mathcal{U}(\mathcal{B}_2(i))|\} \\ &\quad = \sum_{\boldsymbol{x} \in T_V^n(\boldsymbol{y}_1(i))} \Pr[\boldsymbol{x} \in \mathcal{U}(\mathcal{B}_2(i))] \\ &\quad = \sum_{\boldsymbol{x} \in T_V^n(\boldsymbol{y}_1(i))} (1 - \Pr[\boldsymbol{x} \in T_W^n(\boldsymbol{y}_1(i), \boldsymbol{Y}_2(1|i))])^{M_2}. \end{aligned}$$

Using (13),

$$\begin{aligned} E\{|\mathcal{U}(\mathcal{B}_2(i))|\} &\le |T_V^n(\boldsymbol{y}_1(i))| \left(1 - e^{-n[I(W,Q_{2|1}|Q_1)+\epsilon/2]}\right)^{M_2} \\ &\le e^{nH(V|Q_1) - M_2 e^{-n[I(W,Q_{2|1}|Q_1)+\epsilon/2]}}. \end{aligned}$$

We then obtain $E\{|\mathcal{U}(\mathcal{B}_2(i))|\} < 1$ for large enough n by choosing

$$I(W, Q_{2|1}|Q_1) + \frac{3\epsilon}{4} \le \frac{1}{n} \log M_2 \le I(W, Q_{2|1}|Q_1) + \epsilon.$$

The proof is now complete. □

We also present here without proof the converse for joint type covering. The proof will be given in the context of strong covering in Lemma 3. It suffices to replace D_1- and D_2-spheres therein with V- and W-shells.

Lemma 2: For any $n > 0$, if there exist codebooks $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1} \subset T_{Q_1}^n$ and $\{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2} \subset T_{Q_{2|1}}^n(\boldsymbol{y}_1(i))$ which $P_{XY_1Y_2}$-cover T_P^n, then

$$\begin{aligned} \frac{1}{n} \log M_1 + \epsilon(n) &\ge I(Q_1, V) \\ \frac{1}{n} \log M_2 + \epsilon(n) &\ge I(Q_{2|1}, W|Q_1) \end{aligned}$$

where $Q_1, Q_{2|1}, V$, and W are the marginals and conditionals induced by $P_{XY_1Y_2}$, and $\epsilon(n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Lemmas 1 and 2 fully characterize the achievable rate region for joint type covering as the collection of rate pairs greater than $\{I(Q_1, V), I(Q_{2|1}, W|Q_1)\}$. Therefore, were this type of covering used for constructing a scalable coder, the resultant region of achievable rates would be given by $\mathcal{R}'_P(D_1, D_2)$. Since we know that $\mathcal{R}'_P(D_1, D_2) \subset \mathcal{R}_P(D_1, D_2)$ (strict inclusion shown by example in Appendix A), this implies that joint type covering is too strong for scalable source coding purposes, and will result in higher rates.

IV. Achievable Rates in Strong Type Covering

In this section, we will derive the correct characterization of the region of rates that are necessary and sufficient to strongly cover a type. More specifically, we will show that strong (D_1, D_2)-covering of type P with rates R_1 and R_2 is possible if and only if $(R_1, R_2) \in \mathcal{T}_P(D_1, D_2)$, where $\mathcal{T}_P(D_1, D_2)$ consists of rate pairs (R_1, R_2) such that there exists $Q_1 \in \mathcal{M}(\mathcal{Y}_1)$ satisfying

$$I_m(P\|Q_1, D_1) \le R_1 \tag{28}$$

and for all $V \in \mathcal{V}(P, Q_1, D_1)$, there exists $Q_{2|1}(V) \in \mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)$ satisfying

$$I_m(V\|Q_{2|1}(V), Q_1, D_2) \le R_2. \tag{29}$$

Therefore, by definition, we have $\mathcal{T}_P(D_1, D_2) \subseteq \mathcal{R}_P(D_1, D_2)$. On the other hand, we will demonstrate via an example that strict inclusion $\mathcal{T}_P(D_1, D_2) \subset \mathcal{R}_P(D_1, D_2)$ may hold. In particular, the example shows that if the first stage rate is fixed at $R_1 = R_{P,d_1}(D_1)$, i.e., at the value of the rate-distortion function with respect to d_1, evaluated at D_1, then

$$\begin{aligned} &\min\{R_2 : (R_1, R_2) \in \mathcal{T}_P(D_1, D_2)\} \\ &\qquad > \min\{R_2 : (R_1, R_2) \in \mathcal{R}_P(D_1, D_2)\}. \end{aligned}$$

A. Converse for Strong Type Covering

Let us first state and prove a converse result that characterizes necessary conditions for strong type covering. The lemma below essentially proves that there exists no covering strategy that performs better than that which first covers the type with D_1-spheres, and then covers each V-shell in each D_1-sphere with D_2-spheres. Similar ideas can be employed to derive Lemma 2, which characterizes the necessary conditions for joint type covering.

Lemma 3: For any $n > 0$, if there exist codebooks $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1}$ and $\{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2}$ strongly (D_1, D_2)-covering T_P^n, then there exists $Q_1 \in \mathcal{M}(\mathcal{Y}_1)$ such that

$$\frac{1}{n} \log M_1 + \epsilon(n) \ge I_m(P\|Q_1, D_1) \tag{30}$$

and

$$\frac{1}{n} \log M_2 + \epsilon(n) \tag{31}$$

$$\ge \max_{V \in \mathcal{V}(P,Q_1,D_1)} \min_{Q_{2|1} \in \mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)} I_m(V\|Q_{2|1}, Q_1, D_2) \tag{32}$$

where $\epsilon(n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof: Let $f : T_P^n \to \{1,\ldots,M_1\}$ be defined as the smallest $i \in \{1,\ldots,M_1\}$ satisfying $d_1(\boldsymbol{x}, \boldsymbol{y}_1(i)) \le D_1$. Let

$$i^* = \arg\max_i |f^{-1}(i)|$$

and let Q_1 be the type of $\boldsymbol{y}_1(i^*)$. Then since $T_P^n = \bigcup_{i=1}^{M_1} f^{-1}(i)$, we have

$$M_1 \ge \frac{|T_P^n|}{|f^{-1}(i^*)|}. \tag{33}$$

Observe that since $f^{-1}(i^*) \subset T_P^n \cap \mathcal{S}_1(\boldsymbol{y}_1(i^*), D_1)$, and

$$T_P^n \cap \mathcal{S}_1(\boldsymbol{y}_1(i^*), D_1) = \bigcup_{V \in \mathcal{V}^n(P,Q_1,D_1)} T_V^n(\boldsymbol{y}_1(i^*))$$

we have

$$f^{-1}(i^*) = \bigcup_{V \in \mathcal{V}^n(P,Q_1,D_1)} [T_V^n(\boldsymbol{y}_1(i^*)) \cap f^{-1}(i^*)].$$

Therefore,

$$|f^{-1}(i^*)| \le (n+1)^{|\mathcal{X}||\mathcal{Y}_1|} \max_{V \in \mathcal{V}^n(P,Q_1,D_1)} |T_V^n(\boldsymbol{y}_1(i^*))|$$

which, together with (1), (2), and (33), implies

$$\begin{aligned}\frac{1}{n}\log M_1 + |\mathcal{X}|(1+|\mathcal{Y}_1|)\delta(n) &\ge \min_{V \in \mathcal{V}^n(P,Q_1,D_1)} I(Q_1, V)\\ &\ge I_m(P\|Q_1, D_1)\end{aligned}$$

proving (30).

According to the lemma hypothesis

$$T_P^n \cap \mathcal{S}_1(\boldsymbol{y}_1(i^*), D_1) \subset \bigcup_{j=1}^{M_2} \mathcal{S}_2(\boldsymbol{y}_2(j|i^*), D_2)$$

which, in turn, implies that for any $V \in \mathcal{V}^n(P, Q_1, D_1)$

$$T_V^n(\boldsymbol{y}_1(i^*)) = \bigcup_{j=1}^{M_2} [\mathcal{S}_2(\boldsymbol{y}_2(j|i^*), D_2) \cap T_V^n(\boldsymbol{y}_1(i^*))]. \tag{34}$$

Now, observe that $\mathcal{S}_2(\boldsymbol{y}_2(j|i^*), D_2) \cap T_V^n(\boldsymbol{y}_1(i^*))$ is the set of all $\boldsymbol{x}$ which are in some W-shell of $(\boldsymbol{y}_1(i^*), \boldsymbol{y}_2(j|i^*))$ where W is consistent with V and induces a second stage distortion of at most D_2. But if $\boldsymbol{y}_2(j|i^*) \in T_{Q_{2|1}}^n(\boldsymbol{y}_1(i^*))$, this translates to $W \in \mathcal{W}^n(V, Q_{2|1}, Q_1, D_2)$, and therefore we obtain

$$\begin{aligned}&\mathcal{S}_2(\boldsymbol{y}_2(j|i^*), D_2) \cap T_V^n(\boldsymbol{y}_1(i^*))\\ &\qquad = \bigcup_{W \in \mathcal{W}^n(V,Q_{2|1},Q_1,D_2)} T_W^n(\boldsymbol{y}_1(i^*), \boldsymbol{y}_2(j|i^*))\end{aligned} \tag{35}$$

for any $\boldsymbol{y}_2(j|i^*) \in T_{Q_{2|1}}^n(\boldsymbol{y}_1(i^*))$. Using (34) and (35), we can write

$$\begin{aligned}&|T_V^n(\boldsymbol{y}_1(i^*))| \le M_2(n+1)^{|\mathcal{X}||\mathcal{Y}_1||\mathcal{Y}_2|}\\ &\qquad \max_{Q_{2|1} \in \mathcal{C}_{Q_1}^n(\mathcal{Y}_2|\mathcal{Y}_1)} \max_{W \in \mathcal{W}^n(V,Q_{2|1},Q_1,D_2)} |T_W^n|\end{aligned}$$

where, $|T_W^n|$ denotes the size of the W-shell of (any) pair $(\boldsymbol{y}_1 \in T_{Q_1}^n, \boldsymbol{y}_2 \in T_{Q_{2|1}}^n(\boldsymbol{y}_1))$. Hence, using (2), we obtain

$$\begin{aligned}&\frac{1}{n}\log M_2 + |\mathcal{X}||\mathcal{Y}_1|(1+|\mathcal{Y}_2|)\delta(n)\\ &\qquad \ge \min_{Q_{2|1} \in \mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)} I_m(V\|Q_{2|1}, Q_1, D_2)\end{aligned} \tag{36}$$

for all $V \in \mathcal{V}^n(P, Q_1, D_1)$. Equation (31) then follows by first maximizing RHS of (36) over $V \in \mathcal{V}^n(P, Q_1, D_1)$, and then expanding the domain of maximization from $\mathcal{V}^n(P, Q_1, D_1)$ to $\mathcal{V}(P, Q_1, D_1)$ at the expense of another "vanishingly small" term on the left-hand side (LHS). □

B. Achievability of $\mathcal{T}_P(D_1, D_2)$

We now show that rates in $\mathcal{T}_P(D_1, D_2)$ are asymptotically achievable.

Lemma 4: If $(R_1, R_2) \in \mathcal{T}_P(D_1, D_2)$, then there exist codebooks $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1}$ and $\{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2}$ with

$$\begin{aligned}\frac{1}{n}\log M_1 &\le R_1 + \epsilon\\ \frac{1}{n}\log M_2 &\le R_2 + \epsilon\end{aligned}$$

strongly (D_1, D_2)-covering T_P^n, for arbitrary $\epsilon > 0$ and large enough n.

Proof: Using the fact that $\mathcal{M}^n(\mathcal{X}\times\mathcal{Y}_1\times\mathcal{Y}_2)$ is dense in $\mathcal{M}(\mathcal{X}\times\mathcal{Y}_1\times\mathcal{Y}_2)$, and also that $I(Q_1, V)$ and $I(Q_{2|1}, W|Q_1)$ are continuous in all of their parameters, it follows that for large enough n, there must exist $Q_1 \in \mathcal{M}^n(\mathcal{Y}_1)$ such that

$$I_m(P\|Q_1, D_1) \le R_1 + \epsilon/2 \tag{37}$$

and for all $V \in \mathcal{V}^n(P, Q_1, D_1)$, there must exist $Q_{2|1}(V) \in \mathcal{C}_{Q_1}^n(\mathcal{Y}_2|\mathcal{Y}_1)$ such that

$$I_m(V\|Q_{2|1}(V), Q_1, D_1) \le R_2 + \epsilon/2. \tag{38}$$

We will first show for large n the existence of a codebook $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1} \subset T_{Q_1}^n$ such that

$$T_P^n \subseteq \bigcup_{i=1}^{M_1} \mathcal{S}_1(\boldsymbol{y}_1(i), D_1)$$

with

$$\frac{1}{n}\log M_1 \le I_m(P\|Q_1, D_1) + \epsilon/2. \tag{39}$$

Then, for each $i \in \{1,\ldots,M_1\}$ and each $V \in \mathcal{V}^n(P, Q_1, D_1)$, we will show for large n the existence of V-shell refinement codebooks $\{\boldsymbol{y}_2(j, V|i)\}_{j=1}^{M_2(V)} \subset T_{Q_{2|1}(V)}^n$ such that

$$T_V^n(\boldsymbol{y}_1(i)) \subseteq \bigcup_{j=1}^{M_2(V)} \mathcal{S}_2(\boldsymbol{y}_2(j, V|i), D_2)$$

with

$$\frac{1}{n}\log M_2(V) \le I_m(V\|Q_{2|1}(V), Q_1, D_1) + \epsilon/4.$$

Note that since

$$T_P^n \cap \mathcal{S}_1(\boldsymbol{y}_1(i), D_1) = \bigcup_{V \in \mathcal{V}^n(P,Q_1,D_1)} T_V^n(\boldsymbol{y}_1(i))$$

the union

$$\bigcup_{V \in \mathcal{V}^n(P,Q_1,D_1)} \{\boldsymbol{y}_2(j, V|i)\}_{j=1}^{M_2(V)}$$

together with $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1}$, strongly (D_1, D_2)-covers T_P^n. The total number of second-stage codevectors is

$$M_2 = \sum_{V \in \mathcal{V}^n(P,Q_1,D_1)} M_2(V)$$

and

$$\begin{aligned}\frac{1}{n}\log M_2 &\le |\mathcal{X}||\mathcal{Y}_1|\delta(n) + \max_{V \in \mathcal{V}^n(P,Q_1,D_1)} \frac{1}{n}\log M_2(V)\\ &\le \max_{V \in \mathcal{V}^n(P,Q_1,D_1)} I_m(V\|Q_{2|1}(V), Q_1, D_1) + \epsilon/2\end{aligned} \tag{40}$$

for large enough n. The proof will then be complete after combining (39) with (37), and (40) with (38).

Denote by V_0 be the conditional type that achieves $\min_{V\in\mathcal{V}^n(P,Q_1,D_1)} I(Q_1,V)$. The first part of Lemma 1 shows that there exists a codebook $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1}$ that covers T_P^n with V_0-shells such that

$$\frac{1}{n}\log M_1 \le I(Q_1,V_0)+\epsilon/4 \le I_m(P\|Q_1,D_1)+\epsilon/2$$

where the second inequality relies on the fact that $\mathcal{V}^n(P,Q_1,D_1)$ is dense in $\mathcal{V}(P,Q_1,D_1)$. Since $V_0\in\mathcal{V}(P,Q_1,D_1)$, V_0-shells are subsets of D_1-spheres and this codebook also covers T_P^n with D_1-spheres.

Although the first stage codebook guarantees the existence of a codevector in the first stage that has joint type $Q_1\circ V_0$, since we only ask for a D_1-match, the actual joint type of $\boldsymbol{x}$ and its D_1-matching codevector could be any $Q_1\circ V, V\in\mathcal{V}^n(P,Q_1,D_1)$. For every such V, let $W_0(V)$ denote the conditional type that achieves $\min_{W\in\mathcal{W}^n(V,Q_{2|1}(V),Q_1,D_2)} I(Q_{2|1}(V),W|Q_1)$. By the second part of Lemma 1, for every $\boldsymbol{y}_1(i)$ and $V\in\mathcal{V}^n(P,Q_1,D_1)$, there exists a codebook $\{\boldsymbol{y}_2(j,V|i)\}_{j=1}^{M_2(V)}$ that covers $T_V^n(\boldsymbol{y}_1(i))$ with $W_0(V)$-shells such that

$$\begin{aligned}\frac{1}{n}\log M_2(V) &\le I(Q_{2|1}(V),\,W_0(V)|Q_1)+\epsilon/8\\ &\le I_m(V\|Q_{2|1}(V),\,Q_1,D_1)+\epsilon/4\end{aligned}$$

where, again, we use in the second inequality the fact that $\mathcal{W}^n(V,Q_{2|1}(V),Q_1,D_2)$ is dense in $\mathcal{W}(V,Q_{2|1}(V),Q_1,D_2)$. This codebook also covers $T_V^n(\boldsymbol{y}_1(i))$ with D_2-spheres since

$$W_0(V)\in\mathcal{W}^n(V,Q_{2|1}(V),Q_1,D_2).$$

The proof is now complete. □

C. Example: Rates in $\mathcal{R}_P(D_1,D_2)$ may not be Sufficient for Strong Type Covering

Let $\mathcal{X}=\mathcal{Y}_1=\mathcal{Y}_2=\{0,1,2\}$, and $P(x)=1/3$ for all $x\in\mathcal{X}$. Consider the distortion measure given by

$$d(x,y)=\begin{cases}0, & y=x\\ 0.1, & y=x+1 \bmod 3\\ 1, & y=x-1 \bmod 3\end{cases}$$

and let $d_1(\cdot,\cdot)=d_2(\cdot,\cdot)=d(\cdot,\cdot)$. This distortion measure, which is said to be *balanced* since both rows and columns are permutations of a single vector, was analyzed in [9] for its properties regarding successive refinability. There, it was shown that the minimum value

$$\min_{Q_1\in\mathcal{M}(\mathcal{Y}_1)}\ \min_{V\in\mathcal{V}(P,Q_1,D_1)} I(Q_1,V)$$

which corresponds to the nonscalable rate-distortion function $R_{P,d_1}(D_1)$ (cf. (3)), is uniquely[5] achieved by

$$Q_1(y_1)=1/3,\qquad \forall y_1\in\mathcal{Y}_1$$

and

$$V_0(x|y_1)=\frac{1}{1+s+s^{10}}\cdot\begin{cases}1, & x=y_1\\ s^{10}, & x=y_1+1 \bmod 3\\ s, & x=y_1-1 \bmod 3\end{cases}$$

where the parameter $0\le s\le 1$ is determined by the distortion value D_1 according to

$$D_1=\frac{s^{10}+0.1s}{1+s+s^{10}}.$$

We let $D_1=0.06$, which yields $s\approx 0.7121$, and hence

$$V_0(x|y_1)\approx\begin{cases}0.57286, & x=y_1\\ 0.01921, & x=y_1+1 \bmod 3\\ 0.40793, & x=y_1-1 \bmod 3.\end{cases}$$

We also fix

$$R_1=I(Q_1,V_0)=\log 3+\sum_{x\in\mathcal{X}} V_0(x|y_1)\log V_0(x|y_1)\approx 0.33778$$

so that (Q_1,V_0) becomes the unique pair satisfying $I(Q_1,V_0)\le R_1$ (with equality).

Now, $(R_1,R_2)\in\mathcal{R}_P(D_1,D_2)$ if and only if [cf. (4)]

$$\min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)} I_m(V_0\|Q_{2|1},Q_1,D_2)\le R_2.$$

On the other hand, since Q_1 is also the unique choice that satisfies (28), $(R_1,R_2)\in\mathcal{T}_P(D_1,D_2)$ if and only if [cf. (29)]

$$\min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)} I_m(V\|Q_{2|1},Q_1,D_2)\le R_2$$

for *all* $V\in\mathcal{V}(P,Q_1,D_1)$. Therefore, to prove that $\mathcal{T}_P(D_1,D_2)$ is a proper subset of $\mathcal{R}_P(D_1,D_2)$, it suffices to find $V\in\mathcal{V}(P,Q_1,D_1)$ such that

$$\min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)} I_m(V_0\|Q_{2|1},Q_1,D_2) < \min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)} I_m(V\|Q_{2|1},Q_1,D_2).$$

Before proceeding with the demonstration of such V, let us point that the expression

$$\min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)} I_m(V\|Q_{2|1},Q_1,D_2) = \min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)}\ \min_{W\in\mathcal{W}(V,Q_{2|1},Q_1,D_2)} I(Q_{2|1},W|Q_1)$$

corresponds to the rate-distortion function when both the encoder and the decoder have access to side information about the source ([2, Sec. 6.1.1, Case IV], [6]), where the joint distribution of the side information Y_1 and the source X is given by $(Q_1\circ V)(y_1,x)$. We can rewrite this minimization as

$$\min_{Q_{2|1},W}\sum_{y_1\in\mathcal{Y}_1} Q_1(y_1)\cdot\sum_{x\in\mathcal{X}}\sum_{y_2\in\mathcal{Y}_2} Q_{2|1}(y_2|y_1)W(x|y_1,y_2)\log\frac{W(x|y_1,y_2)}{V(x|y_1)}$$

subject to

$$\sum_{y_2\in\mathcal{Y}_2} Q_{2|1}(y_2|y_1)W(x|y_1,y_2)=V(x|y_1)$$

and

$$\sum_{y_1\in\mathcal{Y}_1} Q_1(y_1)\sum_{x\in\mathcal{X}}\sum_{y_2\in\mathcal{Y}_2} Q_{2|1}(y_2|y_1)W(x|y_1,y_2)d_2(x,y_2)\le D_2.$$

[5]The uniqueness of V_0 is generally known (e.g., see [4,, Problem 2.3.3]), whereas that of Q_1 follows from the discussion in [2, Sec. 2.6, Case 1] and the invertibility of the matrix

$$\begin{bmatrix}1 & s & s^{10}\\ s^{10} & 1 & s\\ s & s^{10} & 1\end{bmatrix}$$

for all $s<1$.

We can look at this minimization problem in the context of optimal bit allocation among three sources $V(x|0), V(x|1)$, and $V(x|2)$ subject to the constraint that the average distortion does not exceed D_2. Alternatively, recalling that $Q_1(y_1) = 1/3$, one can recognize the problem as that of rate-distortion optimization for a product source

$$P'(x_0, x_1, x_2) = V(x_0|0)V(x_1|1)V(x_2|2)$$

and a sum distortion measure

$$d((x_0, x_1, x_2),\ (y_{20}, y_{21}, y_{22})) = d_2(x_0, y_{20}) + d_2(x_1, y_{21}) + d_2(x_2, y_{22})$$

as discussed in [2, Section 6.1.1]. It is shown by [2, Theorem 2.8.1] that the rate-distortion curve in this problem is achieved by averaging the rate and distortion coordinates of points at which the individual rate-distortion functions (i.e., for $V(x_0|0), V(x_1|1)$, and $V(x_2|2)$) have equal slopes. When V is a *balanced* backward channel, e.g., $V = V_0$, and the distortion measure is also balanced, e.g., $d_2(x, y_2)$, all such rate-distortion curves will be identical. Hence, one can equivalently compute only one of them via

$$\min_{Q_2 \in \mathcal{M}(\mathcal{Y}_2)} \min_{W' \in \mathcal{V}(P', Q_2, D_2)} I(Q_2, W') \tag{41}$$

where $P'(x) = V(x|0)$. We compute this ordinary rate-distortion curve for the choices of $V = V_0$ and $V = V_1$ (i.e., for $P'(x) = V_0(x|0)$ and $P'(x) = V_1(x|0)$, respectively), where

$$V_1(x|y_1) = \begin{cases} 0.49, & x = y_1 \\ 0.01, & x = y_1 + 1 \bmod 3 \\ 0.5, & x = y_1 - 1 \bmod 3 \end{cases}$$

which is easily verified to be in $\mathcal{V}(P, Q_1, D_1)$. Instead of a painstaking calculation of actual values of (41) for some D_2, one can resort to numerical computation via the Blahut-Arimoto algorithm [3]. Fig. 2 shows the resultant curves for all $0 \leq D_2 \leq 0.06$. Clearly, for a certain range of D_2,

$$\min_{Q_{2|1} \in \mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)} I_m(V_0 \| Q_{2|1}, Q_1, D_2) < \min_{Q_{2|1} \in \mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)} I_m(V_1 \| Q_{2|1}, Q_1, D_2).$$

This, in turn, proves that in this case $\mathcal{T}_P(D_1, D_2) \subset \mathcal{R}_P(D_1, D_2)$.

V. Achievable Rates in Weak Type Covering

In this section, we derive necessary and sufficient rates for weak (D_1, D_2)-covering of types. The following lemma proves that, unlike for strong covering, all the points in the region of achievable rates in scalable source coding are achievable for weak covering.

Lemma 5: If $(R_1, R_2) \in \mathcal{R}_P(D_1, D_2)$, then there exist codebooks $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1}$ and $\{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2}$ with

$$\frac{1}{n} \log M_1 \leq R_1 + \epsilon$$
$$\frac{1}{n} \log M_2 \leq R_2 + \epsilon$$

weakly (D_1, D_2)-covering T_P^n, for arbitrary $\epsilon > 0$ and large enough n.

Proof: It follows from the same arguments as in the Proof of Lemma 4 that there must exist

$$Q_1 \in \mathcal{M}^n(\mathcal{Y}_1)$$
$$Q_{2|1} \in \mathcal{C}_{Q_1}^n(\mathcal{Y}_2|\mathcal{Y}_1)$$
$$V_0 \in \mathcal{V}^n(P, Q_1, D_1)$$

and

$$W_0 \in \mathcal{W}^n(V_0, Q_{2|1}, Q_1, D_2)$$

such that

$$I(Q_1, V_0) \leq R_1 + \epsilon/2$$
$$I(Q_1, V_0) + I(Q_{2|1}, W_0|Q_1) \leq R_1 + R_2 + \epsilon$$

for large enough n. The proof will, therefore, be complete once we show the existence of $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1}$ and $\{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2}$ weakly (D_1, D_2)-covering T_P^n such that

$$\frac{1}{n} \log M_1 \leq I(Q_1, V_0) + \epsilon/2 \tag{42}$$
$$\frac{1}{n} \log M_2 \leq I(Q_{2|1}, W_0|Q_1) + \epsilon/2 \tag{43}$$

as one can always transfer rate to the first layer from the second layer: To transfer a rate of ΔR, simply repeat all the first-layer codevectors $\boldsymbol{y}_1(i)$ $2^{n\Delta R}$ times, uniformly partition each $\{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2}$ into $2^{n\Delta R}$ subcodebooks, and assign each subcodebook to a repeated first-layer codevector. Existence of weakly covering codebooks with rates as in (42) and (43) follows immediately from Lemma 1 and the fact that a $P_{XY_1Y_2}$-covering with $V \in \mathcal{V}^n(P, Q_1, D_1)$ and $W \in \mathcal{W}^n(V, Q_{2|1}, Q_1, D_2)$ is also a weak (D_1, D_2)-covering of T_P^n. □

Now, we turn to the converse result which states that, for weak (D_1, D_2)-covering, $\mathcal{R}_P(D_1, D_2)$ characterizes the necessary rates as well.

Lemma 6: For any $n > 0$, if there exist codebooks $\{\boldsymbol{y}_1(i)\}_{i=1}^{M_1}$ and $\{\boldsymbol{y}_2(j|i)\}_{j=1}^{M_2}$ weakly (D_1, D_2)-covering T_P^n, then

$$\left(\frac{1}{n} \log M_1 + \epsilon(n), \frac{1}{n} \log M_2 + \epsilon(n)\right) \in \mathcal{R}_P(D_1, D_2),$$

where $\epsilon(n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof: The proof is almost identical to that of ([13] Lemma 1). The only difference here is that, the hypothesis of the lemma claims the existence of a weak (D_1, D_2)-covering of only type P, instead of that of the high probability set around T_P^n (i.e., the *typical* set.) □

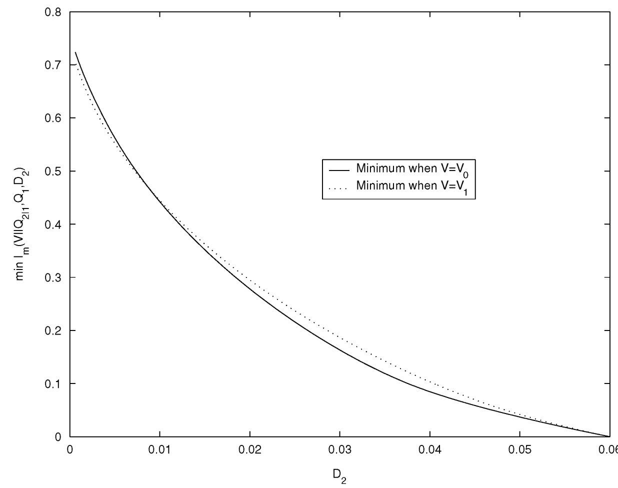

Fig. 2. Comparison of $\min_{Q_{2|1} \in \mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)} I_m(V_0 \| Q_{2|1}, Q_1, D_2)$ with $\min_{Q_{2|1} \in \mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)} I_m(V_1 \| Q_{2|1}, Q_1, D_2)$. Clearly, the latter is strictly larger for a certain range of D_2.

VI. Reinvestigation of Guessing Exponents

In [12], the authors showed that a guessing exponent of

$$\max_{P'\in\mathcal{M}(\mathcal{X})}[\rho K(D_1,D_2,P')-\mathcal{D}(P'\|P)]$$

is achievable through strong (D_1,D_2)-covering of the type $P_{\boldsymbol{x}}$, which was assumed to be known beforehand. However, that result needs to be restated so as to reflect the true region of achievable rates derived in Section IV. The achievable ρth-order guessing exponent obtained through strong type covering must be stated as

$$\max_{P'\in\mathcal{M}(\mathcal{X})}[\rho L(D_1,D_2,P')-\mathcal{D}(P'\|P)]$$

where

$$L(D_1,D_2,P)\triangleq\min_{Q_1\in\mathcal{M}(\mathcal{Y}_1)}\max\Big\{I_m(P\|Q_1,D_1),\max_{V\in\mathcal{V}(P,Q_1,D_1)}\min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)}I_m(V\|Q_{2|1},Q_1,D_2)\Big\}. \tag{44}$$

We now formally prove that $L(D_1,D_2,P)\geq K(D_1,D_2,P)$, which does not follow immediately from (44).

Lemma 7: $L(D_1,D_2,P)\geq K(D_1,D_2,P)$ for all $P\in\mathcal{M}(\mathcal{X})$.

Proof: Let

$$L(D_1,D_2,P|Q_1)\triangleq\max\Big\{I_m(P\|Q_1,D_1),\max_{V\in\mathcal{V}(P,Q_1,D_1)}\min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)}I_m(V\|Q_{2|1},Q_1,D_2)\Big\} \tag{45}$$

and

$$\begin{aligned}K(D_1,D_2,P|Q_1)&\triangleq\min_{\substack{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)\\ V\in\mathcal{V}(P,Q_1,D_1)\\ W\in\mathcal{W}(V,Q_{2|1},Q_1,D_2)}}\max\Big\{I(Q_1,V),I(Q_{2|1},W|Q_1)\Big\}\\&=\min_{V\in\mathcal{V}(P,Q_1,D_1)}\max\Big\{I(Q_1,V),\min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)}I_m(V\|Q_{2|1},Q_1,D_2)\Big\}\end{aligned}$$

where the latter equality follows from the general identity

$$\min_x\max\{c,f(x)\}=\max\{c,\min_x f(x)\}$$

when c is a constant. It suffices to show

$$L(D_1,D_2,P|Q_1)\geq K(D_1,D_2,P|Q_1)$$

because then

$$\min_{Q_1\in\mathcal{M}(\mathcal{Y}_1)}L(D_1,D_2,P|Q_1)\geq\min_{Q_1\in\mathcal{M}(\mathcal{Y}_1)}K(D_1,D_2,P|Q_1)$$

proving the lemma.

If $L(D_1,D_2,P|Q_1)=I_m(P\|Q_1,D_1)$, then for all $V\in\mathcal{V}(P,Q_1,D_1)$

$$\begin{aligned}I(Q_1,V)&\geq L(D_1,D_2,P|Q_1)\\&\geq\min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)}I_m(V\|Q_{2|1},Q_1,D_2)\end{aligned}$$

which implies

$$\begin{aligned}K(D_1,D_2,P|Q_1)&=\min_{V\in\mathcal{V}(P,Q_1,D_1)}I(Q_1,V)\\&=I_m(P\|Q_1,D_1)\\&=L(D_1,D_2,P|Q_1).\end{aligned}$$

On the other hand, if

$$L(D_1,D_2,P|Q_1)=\max_{V\in\mathcal{V}(P,Q_1,D_1)}\min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)}I_m(V\|Q_{2|1},Q_1,D_2)$$

then for any $V\in\mathcal{V}(P,Q_1,D_1)$

$$\begin{aligned}K(D_1,D_2,P|Q_1)&\leq\max\Big\{I(Q_1,V),\min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)}I_m(V\|Q_{2|1},Q_1,D_2)\Big\}\\&\leq\max\Big\{I(Q_1,V),L(D_1,D_2,P|Q_1)\Big\}.\end{aligned}$$

Choosing V such that

$$I(Q_1,V)=I_m(P\|Q_1,D_1)\leq L(D_1,D_2,P|Q_1)$$

we obtain $K(D_1,D_2,P|Q_1)\leq L(D_1,D_2,P|Q_1)$. □

Now, based on the example in Section IV-C, we will demonstrate that there exist cases where $L(D_1,D_2,P)>K(D_1,D_2,P)$. To that end, we lower bound $K(D_1,D_2,P)$ as follows:

$$\begin{aligned}K(D_1,D_2,P)&\geq\min_{\substack{Q_1\in\mathcal{M}(\mathcal{Y}_1)\\ Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)\\ V\in\mathcal{V}(P,Q_1,D_1)\\ W\in\mathcal{W}(V,Q_{2|1},Q_1,D_2)}}I(Q_1,V)\\&=\min_{\substack{Q_1\in\mathcal{M}(\mathcal{Y}_1)\\ V\in\mathcal{V}(P,Q_1,D_1)}}I(Q_1,V)\\&=R_{P,d_1}(D_1).\end{aligned}$$

This inequality is satisfied with equality if and only if for all (Q_1,V) achieving $R_{P,d_1}(D_1)$, there exists $Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)$ and $W\in\mathcal{W}(V,Q_{2|1},Q_1,D_2)$ such that $I(Q_{2|1},W|Q_1)\leq R_{P,d_1}(D_1)$, i.e.,

$$\min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)}I_m(V\|Q_{2|1},Q_1,D_2)\leq R_{P,d_1}(D_1). \tag{46}$$

Recall that in the example in Section IV-C, we set $D_1=0.06$ so that there is a unique pair of (Q_1,V) achieving $I(Q_1,V)=R_{P,d_1}(D_1)\approx 0.33778$ (i.e., $Q_1(y_1)=1/3$ and $V=V_0$). From Fig. 3, which is a closeup of Fig. 2, we can deduce that when $D_2=0.016$, (46) is satisfied for this (Q_1,V). Therefore, $K(D_1,D_2,P)=R_{P,d_1}(D_1)$ when $D_1=0.06$ and $D_2=0.016$. Further, the minimum in

$$K(D_1,D_2,P)=\min_{Q_1\in\mathcal{M}(\mathcal{Y}_1)}K(D_1,D_2,P|Q_1)$$

is also uniquely achieved by the same Q_1. Since $L(D_1,D_2,P|Q_1)\geq K(D_1,D_2,P|Q_1)$ for all $Q_1\in\mathcal{M}(\mathcal{Y}_1)$, it then suffices to show

$$L(D_1,D_2,P|Q_1)>R_{P,d_1}(D_1)$$

for $Q_1(y_1)=1/3$ in order to show $L(D_1,D_2,P)>K(D_1,D_2,P)$. But from (45), this implies either

$$I_m(P\|Q_1,D_1)>R_{P,d_1}(D_1)$$

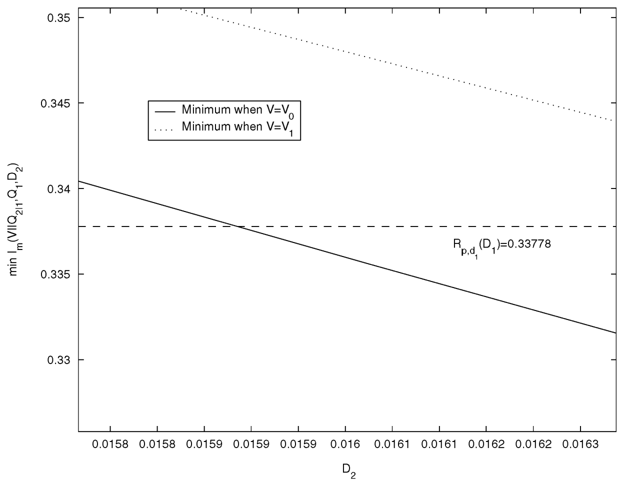

Fig. 3. Closeup of Fig. 2 around $D_2 = 0.016$.

which is impossible (since $I_m(P\|Q_1, D_1) = R_{P,d_1}(D_1)$), or

$$\min_{Q_{2|1}\in\mathcal{C}(\mathcal{Y}_2|\mathcal{Y}_1)} I_m(V\|Q_{2|1}, Q_1, D_2) > R_{P,d_1}(D_1)$$

for *some* $V \in \mathcal{V}(P, Q_1, D_1)$. From Fig. 3, we observe that the latter is true for $D_2 = 0.016$ and $V = V_1$, proving that, in general $L(D_1, D_2, P)$ is larger than $K(D_1, D_2, P)$.

The broader problem of proving the existence of cases where

$$\max_{P'\in\mathcal{M}(\mathcal{X})}[\rho L(D_1, D_2, P') - \mathcal{D}(P'\|P)] > \max_{P'\in\mathcal{M}(\mathcal{X})}[\rho K(D_1, D_2, P') - \mathcal{D}(P'\|P)]$$

remains open even after the demonstration of $L(D_1, D_2, P) > K(D_1, D_2, P)$. More specifically, if the optimal P' in (22) satisfies $L(D_1, D_2, P') = K(D_1, D_2, P')$, then the two maxima shown become identical. However, it will be extremely surprising if both maxima are not achieved by the uniform distribution, e.g., by $P(x) = 1/3$ in our example in Section IV-C, when the distortion measures are balanced. If, indeed, the uniform distribution achieves the maximum on the RHS, then the inequality immediately follows from the discussion.

In a related problem, where the data $\boldsymbol{x}$ is not created by a DMS, but is randomly drawn with a uniform distribution over a *known* type class T_P^n, then with our modification in Section II-E, the converse in [12] naturally extends to

$$\liminf_{n\longrightarrow\infty} \frac{1}{n} \min_{\mathcal{G}_1,\mathcal{G}_2} \log E\{G(\boldsymbol{X})^\rho\} \geq \rho J(D_1, D_2, P).$$

However, the strong type covering strategy will not achieve this lower bound, as we have proven that the achieved exponent in that case is given instead by $\rho L(D_1, D_2, P)$. Furthermore, even if (18) is truly a converse,[6] implying $\rho K(D_1, D_2, P)$ as a converse for the scenario under consideration, there still remains a gap between the achievable and converse guessing exponents.

Let us now consider a modification of the hierarchical guessing problem, where the goal is still guessing data $\boldsymbol{x}$ drawn with a uniform distribution over a known type class T_P^n within distortion (D_1, D_2), but Bob can inform Alice whether her guesses are of a certain prespecified joint-type $P_{XY_1Y_2}$ with his observed vector: Alice presents Bob with a sequence of guesses from $\mathcal{Y}_1^n$ till Bob declares he has found a guess $\boldsymbol{y}_1(i)$ whose joint type with his observation is P_{XY_1}. Alice then presents Bob with a another sequence of guesses from $\mathcal{Y}_2^n$ depending on the winning index in the first round i till Bob finds a guess $\boldsymbol{y}_2(j|i)$ such that the joint type of $(\boldsymbol{x}, \boldsymbol{y}_1(i), \boldsymbol{y}_2(j|i))$ is $P_{XY_1Y_2}$. Suppose the joint type being checked is $P^*_{XY_1Y_2}$, the distribution that achieves $K(D_1, D_2, P)$, then the joint type covering lemma for $P^*_{XY_1Y_2}$ implies that $\rho K(D_1, D_2, P)$ is achievable, unlike in the original problem.

[6]Recall that we only invalidated the current proof of (18). In particular, we did not show an example where we achieve a guessing exponent lower than the RHS of (18).

APPENDIX A

In this Appendix, we prove that $\mathcal{R}'_P(D_1, D_2) \neq \mathcal{R}_P(D_1, D_2)$ in general.

Let the source and reproduction alphabets be $\mathcal{X} = \{0, 1, 2, 3\}$ and $\mathcal{Y}_1 = \mathcal{Y}_2 = \{a, b\}$. Also let $P = \{p_0, p_1, p_2, p_3\}$ and

$d_1(x, y_1) =$

	a	b
0	0	1
1	1	0
2	1	0
3	0	1

and

$d_2(x, y_2) =$

	a	b
0	0	1
1	0	1
2	1	0
3	1	0

.

If we set $D_1 = D_2 = 0$, the only $(Q_1, Q_{2|1}, V, W)$-quadruple satisfying the consistency and distortion constraints $W \in \mathcal{W}(V, Q_{2|1}, Q_1, D_2)$ and $V \in \mathcal{V}(P, Q_1, D_1)$ is given by

$$Q_1(y_1) = \begin{cases} p_0 + p_3 & y_1 = \mathrm{a} \\ p_1 + p_2 & y_1 = \mathrm{b} \end{cases}$$

$$Q_{2|1}(y_2|y_1) = \begin{cases} \frac{p_0}{p_0+p_3} & y_1 = \mathrm{a}, y_2 = \mathrm{a} \\ \frac{p_3}{p_0+p_3} & y_1 = \mathrm{a}, y_2 = \mathrm{b} \\ \frac{p_1}{p_1+p_2} & y_1 = \mathrm{b}, y_2 = \mathrm{a} \\ \frac{p_2}{p_1+p_2} & y_1 = \mathrm{b}, y_2 = \mathrm{b} \end{cases}$$

$V(x|y_1) =$

	a	b
0	$\frac{p_0}{p_0+p_3}$	0
1	0	$\frac{p_1}{p_1+p_2}$
2	0	$\frac{p_2}{p_1+p_2}$
3	$\frac{p_3}{p_0+p_3}$	0

$W(x|y_1, y_2) =$

	aa	ab	ba	bb
0	1	0	0	0
1	0	0	1	0
2	0	0	0	1
3	0	1	0	0

from which we can deduce

$$I(Q_1, V) = H(Q_1) \tag{47}$$

$$I(Q_{2|1}, W|Q_1) = H(V|Q_1) = H(P) - H(Q_1) \tag{48}$$

$$\frac{I(Q_1, V) + I(Q_{2|1}, W|Q_1)}{2} = \frac{H(P)}{2}. \tag{49}$$

Choosing the uniform distribution $p_0 = p_1 = p_2 = p_3 = 1/4$ in (47) and (48) yields $I(Q_1, V) = I(Q_{2|1}, W|Q_1) = \log 2$, implying that $(R_1 = 2\log 2, R_2 = 0)$ falls inside of $\mathcal{R}_P(D_1, D_2)$ but outside of $\mathcal{R}'_P(D_1, D_2)$.

APPENDIX B

Recall that since $K(D_1, D_2, P) \geq J(D_1, D_2, P)$, the RHS of (18) is greater than or equal to that of (20). To eliminate the remote possibility that these two guessing exponent converses might in fact be identical, we construct a counterexample where the inequality is strict. Toward this end, we use the same setting in Appendix A, but set $p_0 = p_3 = 0.1$ and $p_1 = p_2 = 0.4$.

Now, let us fix $\rho = 1$ and evaluate

$$\begin{aligned}
&\max_{P' \in \mathcal{M}(\mathcal{X})} [K(D_1, D_2, P') - \mathcal{D}(P' \| P)] \\
&= \max_{P' \in \mathcal{M}(\mathcal{X})} \left[\max\{f(p'_0, p'_1, p'_2, p'_3), g(p'_0, p'_1, p'_2, p'_3)\} - \sum_i p'_i \log \frac{p'_i}{p_i} \right] \\
&= \max_{P' \in \mathcal{M}(\mathcal{X})} \left[\max \left\{ f(p'_0, p'_1, p'_2, p'_3) - \sum_i p'_i \log \frac{p'_i}{p_i}, g(p'_0, p'_1, p'_2, p'_3) - \sum_i p'_i \log \frac{p'_i}{p_i} \right\} \right] \\
&= \max \left\{ \max_{P' \in \mathcal{M}(\mathcal{X})} \left[f(p'_0, p'_1, p'_2, p'_3) - \sum_i p'_i \log \frac{p'_i}{p_i} \right], \max_{P' \in \mathcal{M}(\mathcal{X})} \left[g(p'_0, p'_1, p'_2, p'_3) - \sum_i p'_i \log \frac{p'_i}{p_i} \right] \right\}
\end{aligned}$$

where we employ (47) and (48) in defining f and g as

$$\begin{aligned}
f(p'_0, p'_1, p'_2, p'_3) = &-(p'_0 + p'_3)\log(p'_0 + p'_3) \\
&- (p'_1 + p'_2)\log(p'_1 + p'_2)
\end{aligned}$$

and

$$\begin{aligned}
g(p'_0, p'_1, p'_2, p'_3) = &-\sum_i p'_i \log p'_i + (p'_0 + p'_3)\log(p'_0 + p'_3) \\
&+ (p'_1 + p'_2)\log(p'_1 + p'_2).
\end{aligned}$$

We will easily evaluate

$$\max_{P' \in \mathcal{M}(\mathcal{X})} \left[f(p'_0, p'_1, p'_2, p'_3) - \sum_i p'_i \log \frac{p'_i}{p_i} \right]$$

since the argument is a concave function of P'. However, evaluation of the second maximum could be more cumbersome. We recourse to lower bounding that maximum, thereby lower bounding the overall guessing exponent.

For the first maximum, forming the Lagrangian as

$$f(p'_0, p'_1, p'_2, p'_3) - \sum_i p'_i \log \frac{p'_i}{p_i} + \lambda \left(\sum_i p'_i \right)$$

and setting its derivative w.r.t. p'_0 to zero, we obtain

$$\log \frac{p'_0(p'_0 + p'_3)}{p_0} = \lambda - 2$$

or

$$p'_0(p'_0 + p'_3) = \lambda' p_0.$$

Repeating the same for p'_1, p'_2, and p'_3 yields

$$\begin{aligned}
p'_1(p'_1 + p'_2) &= \lambda' p_1 \\
p'_2(p'_1 + p'_2) &= \lambda' p_2 \\
p'_3(p'_0 + p'_3) &= \lambda' p_3.
\end{aligned}$$

One can then extract the solution to be $p'_0 = p'_3 = \frac{1}{6}$ and $p'_1 = p'_2 = \frac{1}{3}$. Therefore,

$$\begin{aligned}
\max_{P' \in \mathcal{M}(\mathcal{X})} \left[f(p'_0, p'_1, p'_2, p'_3) - \sum_i p'_i \log \frac{p'_i}{p_i} \right] &= 2\log 3 - \log 5 \\
&\approx 0.5878.
\end{aligned}$$

For the second maximum, we choose $p'_0 = p'_3 = 0.1$ and $p'_1 = p'_2 = 0.4$, and obtain

$$\max_{P' \in \mathcal{M}(\mathcal{X})} \left[g(p'_0, p'_1, p'_2, p'_3) - \sum_i p'_i \log \frac{p'_i}{p_i} \right] \geq \log 2 \approx 0.6931.$$

Therefore, we have

$$\max_{P' \in \mathcal{M}(\mathcal{X})} [K(D_1, D_2, P') - \mathcal{D}(P' \| P)] \geq 0.6931. \tag{50}$$

We now turn to the evaluation of

$$\begin{aligned}
&\max_{P' \in \mathcal{M}(\mathcal{X})} [J(D_1, D_2, P') - \mathcal{D}(P' \| P)] \\
&= \max \left\{ \max_{P' \in \mathcal{M}(\mathcal{X})} \left[f(p'_0, p'_1, p'_2, p'_3) - \sum_i p'_i \log \frac{p'_i}{p_i} \right], \max_{P' \in \mathcal{M}(\mathcal{X})} \left[h(p'_0, p'_1, p'_2, p'_3) - \sum_i p'_i \log \frac{p'_i}{p_i} \right] \right\}
\end{aligned}$$

where, using (49)

$$h(p'_0, p'_1, p'_2, p'_3) = -\frac{1}{2} \sum_i p'_i \log p'_i.$$

We already evaluated the first maximum to be $2\log 3 - \log 5$. The second maximum is also easily found since its argument is concave in P'. Using the same Lagrangian analysis as above, we obtain

$$\begin{aligned}
(p'_0)^{\frac{3}{2}} &= \lambda' p_0 \\
(p'_1)^{\frac{3}{2}} &= \lambda' p_1 \\
(p'_2)^{\frac{3}{2}} &= \lambda' p_2 \\
(p'_3)^{\frac{3}{2}} &= \lambda' p_3.
\end{aligned}$$

The solution then becomes

$$p'_0 = p'_3 = \frac{1}{2(1 + 16^{\frac{1}{3}})}$$

and

$$p'_1 = p'_2 = \frac{16^{\frac{1}{3}}}{2(1 + 16^{\frac{1}{3}})}$$

yielding

$$\max_{P' \in \mathcal{M}(\mathcal{X})} \left[h(p'_0, p'_1, p'_2, p'_3) - \sum_i p'_i \log \frac{p'_i}{p_i} \right] \approx 0.6248$$

and therefore

$$\begin{aligned}
\max_{P' \in \mathcal{M}(\mathcal{X})} [J(D_1, D_2, P') - \mathcal{D}(P' \| P)] &\approx \max\{0.5878, 0.6248\} \\
&= 0.6248. \qquad (51)
\end{aligned}$$

Comparing (50) and (51) yields the desired result.

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Computing the Channel Capacity and Rate-Distortion Function With Two-Sided State Information

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***Abstract*—In this correspondence, we present iterative algorithms that numerically compute the capacity-power and rate-distortion functions for coding with two-sided state information. Numerical examples are provided to demonstrate efficiency of our algorithms.**

***Index Terms*—Blahut–Arimoto algorithm, channel capacity, coding with side information, Gel'fand–Pinsker problem, rate-distortion function, Wyner–Ziv problem.**

I. INTRODUCTION

Coding with side information has gained increased research interest recently due to its great practical potentials. For example, source coding with side information at the decoder (a.k.a. Wyner–Ziv coding [1]) is recognized as an important component in emerging wireless sensor networks; on the other hand, channel coding with side information at the encoder (a.k.a. Gel'fand–Pinsker coding [2]) can be used to model the digital watermarking problem [3] and also applies to broadcast channel coding [4]. However, very often, it is necessary to use a more general setup with two-sided state information where both the encoder and the decoder have the access to (possibly different) side information. The capacity-power and the rate-distortion functions in this case are given by [5]

$$C(P)=\max_{q'(x|u,s_1)q(u|s_1):E[\mathfrak{p}(S_1,S_2,X)]\leq P} I(U;Y,S_2)-I(U;S_1) \quad (1)$$

and

$$R(D)=\min_{q(u|s_1,x)q'(\hat{x}|s_2,u):E[\mathfrak{d}(X,\hat{X})]\leq D} I(U;X,S_1)-I(U;S_2) \quad (2)$$

respectively, where independent and identically distributed (i.i.d.) random variables X and Y are the channel input and output in the channel coding problem, X and $\hat{X}$ are the source input and the reconstructed output in the source coding problem, S_1 and S_2 are side information at the encoders and the decoders, respectively, and U is an auxiliary random variable. P and D are the power and distortion constraints for the respective channel coding and source coding problems with $\mathfrak{p}(\cdot,\cdot,\cdot)$ and $\mathfrak{d}(\cdot,\cdot)$ being the power and distortion measures. The expressions in both (1) and (2) are optimized over valid conditional probability mass functions (PMFs) $q(\cdot|\cdot)$ and $q'(\cdot|\cdot)$.

Calculations of capacity-power and rate-distortion functions are difficult optimization problems. For conventional source and channel coding, Blahut–Arimoto algorithms [6], [7] provide efficient numerical solutions for memoryless channels and general i.i.d. sources with arbitrary power and distortion measures. These optimization techniques were later generalized in [8]. Extensions to channels and sources with memory were given in [9] and [10].

However, when side information is present at the encoder and/or the decoder, calculation of channel capacity and rate-distortion function

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