An Achievable Rate Region for Distributed Source Coding and Dispersive Information Routing

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***Abstract*—This paper considers the problem of optimal multi-hop routing of correlated sources over a network with multiple sinks and arbitrary network demands. We recently introduced a new routing paradigm in [10] called 'dispersive information routing' (DIR), wherein the intermediate nodes are allowed to split a packet and forward a subset of the received bits on each forward path. DIR ensures that each sink receives just the information it requires to decode the sources it intends to reconstruct, and thereby outperforms conventional routing techniques in the literature. We proposed an encoding scheme called 'power binning' which achieves complete rate region and the minimum cost under this paradigm when each sink is allowed to receive packets only from the sources it wants to reconstruct. This paper considers the optimum encoding scheme when every source can (possibly) communicate with every sink irrespective of what the sinks reconstruct. This generalization happens to be considerably more complex and we derive an achievable rate region and an associated achievable cost using principles from distributed source coding and multiple descriptions encoding.**

***Index Terms*—Distributed source coding, joint compression and routing**

I. Introduction

The problem of minimum cost routing of correlated sources over a multi-hop network has recently attracted researchers due to its direct applicability in sensor networks. What makes this problem particularly interesting is the the challenge of designing encoders which address the interplay between joint compression and routing. Broadly, the research in this field can be grouped into two camps. The first approach performs compression at intermediate nodes [1] and the second resorts to distributed source coding [2], [3], [4]. This paper focuses on the latter category.

The field of distributed source coding (multi-terminal source coding) began with the seminal work of Slepian and Wolf [5] and Wyner and Ziv [6]. Several publications followed considering different multi-terminal scenarios and obtaining achievability bounds for them [7]. Han and Kobayashi [8] (see also [9]) derived an achievable rate region for a general multi-terminal source coding problem with multiple sources and sinks, with each source being reconstructed at a subset of sinks losslessly.

One of the first attempts to unify Slepian-Wolf compression and routing in a network was by Cristescu et.al in [2]. Here, we call the routing mechanism they considered as 'Broadcasting' [1], wherein each source broadcasts its information to all sinks which intend to reconstruct it (which is motivated by routing mechanisms for independent sources). [3] showed that Slepian-Wolf compression followed by 'Broadcasting' is optimum for single sink networks and cannot be outperformed by any other joint compression-routing scheme. However, [4] demonstrated the extent of suboptimality of 'Broadcasting' in case of multi-sink networks.

In a precursor work [10], we introduced a new routing paradigm called 'dispersive information routing' (DIR), wherein the intermediate nodes are allowed to 'split' a packet and forward different subsets of the packets on each forward path. It was demonstrated using simple examples that DIR outperforms broadcasting for multi-sink networks. In [10], we considered the scenario where sinks receive packets only from sources which they intend to reconstruct and derived the complete rate region and the minimum cost achievable under DIR. This scenario is called 'no helpers' case in the literature [9]. In this paper, we consider the more general case wherein each sink can (possibly) receive packets from any source immaterial of what they reconstruct. This scenario was recently addressed for broadcasting in [4] and was shown to have substantial gains over the 'no helper case', albeit being considerably more complex. Here, we derive an achievable rate region for the general DIR problem with helpers using principles from Multiple-Descriptions encoding [11] and Han and Kobayashi decoding.

In the remaining part of this section, we motivate DIR using an example where there are no helpers. In section II, we illustrate the new coding scheme using one of the simplest scenarios with helpers and extend it to a general network in section III.

A. Motivating example

Consider the network shown in Figure 1a. A source X_0 is to be reconstructed at two sinks S_1 and S_2 which have access to side information X_1 and X_2 respectively. Source X_0 communicates with the two sinks through an intermediate node (we term the 'collector') which is functionally a simple router. The edge weights on each path in the network is as shown in the Figure. Assume that the cost of communication through a link is a simple product of the rate and the edge

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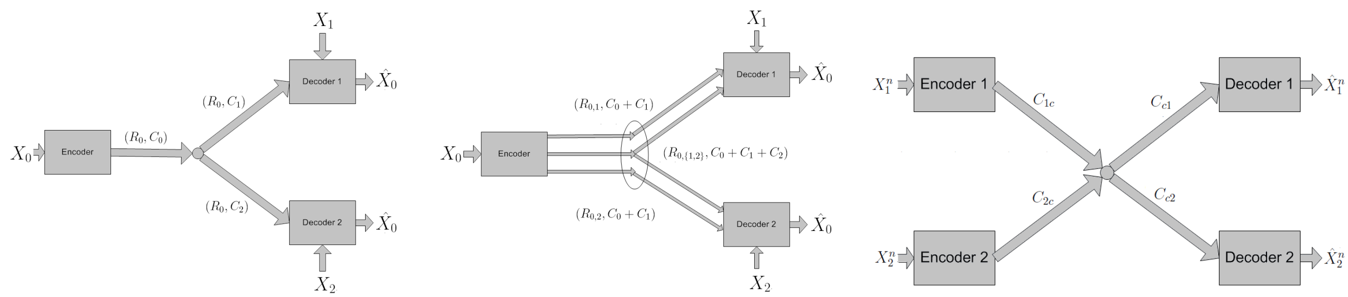
[1]Note that we loosely use the term broadcasting instead of multi-casting to stress the fact that *all* the bits sent by a source are routed to all the sinks which reconstruct it



(a) Broadcast Routing (b) DIR (c) 2 Source - 2 Sink example

Figure 1: (a) Rates and weights for 'Broadcasting' (b) Rates and effective weights for 'DIR' (c) The 2 Source - 2 Sink example. Each source acts as the principle source for one sink and as a helper for the other.

weight, i.e. $f(r, c) = rc$. The objective is to find the minimum communication cost for lossless reconstruction of X_0 at the two sinks.

Under broadcast routing [2], the collector forwards all the bits it receives to both the sinks. The minimum rate at which the source sends for lossless reconstruction at both the sinks is $R = \max(H(X_0|X_1), H(X_0|X_2))$ and hence, the minimum communication cost under broadcast routing is $C_B = (C_0 + C_1 + C_2)\max(H(X_0|X_1), H(X_0|X_2))$. However, observe that the minimum rates required on the branches connecting the collector to sinks S_1 and S_2 are $H(X_0|X_1)$ and $H(X_0|X_2)$ respectively. Clearly, broadcast routing leads to suboptimality on one of these two branches if $H(X_0|X_1) \neq H(X_0|X_2)$.

We introduced a new routing technique in [10], called dispersive information routing (DIR), which achieves optimal cost for this network. Under this new paradigm, the intermediate nodes (collector in this example) are allowed to 'split' a packet and forward different subsets of the received bits on the forward paths. We could equivalently think of the source transmitting 3 smaller packets to the collector. The first two packets, at rates R_1 and R_2, are destined to sinks S_1 and S_2 respectively and the third packet at rate R_{12} is destined to both the sinks, as shown in Figure 1b. We demonstrated using a simple variant of the random binning paradigm, called 'Power binning', that the complete achievable rate region for the tuple (R_1, R_2, R_{12}) is:

$$\begin{aligned} R_1 + R_{12} &\geq H(X_0|X_1) \\ R_2 + R_{12} &\geq H(X_0|X_2) \end{aligned} \tag{1}$$

The minimum cost operating point satisfies equations 1 and minimizes the cost function $C_{DIR} = \{(C_0 + C_1)R_1 + (C_0 + C_2)R_2 + (C_0 + C_1 + C_2)R_{12}\}$. The solution is either of the two points (R_1, R_2, R_{12})=$\{0, H(X_0|X_1), H(X_0|X_2) - H(X_0|X_1)\}$ or $\{H(X_0|X_1) - H(X_0|X_2), H(X_0|X_2), 0\}$ and both achieve a lower cost compared to broadcast routing if $H(X_0|X_1) \neq H(X_0|X_2)$. This example clearly demonstrates the gains of DIR over broadcast routing to communicate correlated sources over a network.

For a general network with K sinks, each source transmits 2^K packets and each packet is routed to a subset of sinks. In [10], we showed that, when only the requested sources (sources being reconstructed) are allowed to communicate with the sinks, power binning achieves the complete rate region and hence, achieves the minimum communication cost under DIR. The extent of suboptimality due to this restriction was investigated in [4], where examples were presented to show potentially unbounded gains by allowing unrequested sources to send information to a sink in the context of 'broadcasting'. In this paper, we find an achievable rate region and an associated cost for dispersive information routing with helpers.

We note that there has been considerable volume of work related to minimum cost network coding for correlated sources [12]. Note that unlike network coding [12], DIR does not require possibly expensive coders at intermediate nodes, but rather can always be realized using conventional routers with each source transmitting multiple packets into the network intended to different subsets of sinks. Therefore, hereafter, we interchangeably use the ideas of "packet splitting" at intermediate nodes and conventional routing of smaller packets, noting that either can be realized using the other. The potential implications of DIR on network coding are beyond the scope of this paper and will be considered as part of future work.

II. 2 Source - 2 Sink Example

To keep the notations and understanding simple, we begin with one of the simplest setups which illustrates the underlying ideas. We will provide intuitive description for the encoding scheme here and defer the formal proofs for the general case as part of Theorem 1 in section III. Consider the network shown in Figure 1c. Two sources s_1 and s_2 observe correlated random variables X_1 and X_2. Two sinks S_1 and S_2 require lossless reconstructions of X_1 and X_2 respectively. The sources can communicate with the sinks only through a collector node. The edge weights are as shown in the figure. Observe that, while each source is requested by one sink, they act as helpers for the other.

Under dispersive information routing, each source transmits a packet to every subset of sinks. In this example, source s_1 sends 3 packets to the collector at rates $(R_{1,1}, R_{1,2}, R_{1,12})$ respectively. The collector forwards the first packet to sink S_1, the second to S_2 and the third to both S_1 and S_2.

Similarly, source s_2 sends 3 packets to the collector at rates $(R_{2,1}, R_{2,2}, R_{2,12})$ which are forwarded to the corresponding sinks. Note that, the rates of some of these packets could be 0. Our objective is to determine the set of achievable rate tuples $(R_{1,1}, R_{1,2}, R_{1,12}, R_{2,1}, R_{2,2}, R_{2,12})$ which allow for lossless reconstruction of respective sources at the two sinks. The minimum cost then follows by finding the point in the achievable rate region which minimizes the effective communication cost:

$$\begin{aligned} C_{DIR} &= \sum_{i=1}^{2}(C_{ic}+C_{ci})R_{i,i} + (C_{1c}+C_{c2})R_{1,2} + \\ &\quad \sum_{i=1}^{2}(C_{ic}+C_{c1}+C_{c2})R_{i,12} + (C_{2c}+C_{c1})R_{2,1} \end{aligned} \tag{2}$$

A non-single letter characterization of the complete rate region is possible using the results of Han and Kobayashi in [8]. They also provide a single-letter partial achievable rate region. However, their result assumes 3 independent encoders at each source, which is an unnecessary restriction. We present a more general rate region, which maintains the dependencies between the messages at each encoder.

Suppose we are given random variables $(U_{1,12}, U_{1,1}, U_{1,2}, U_{2,12}, U_{2,1}, U_{2,2})$ jointly distributed with (X_1, X_2) such that the following Markov chain conditions hold:

$$\begin{aligned} (U_{1,12}, U_{1,1}) &\leftrightarrow X_1 \leftrightarrow X_2 \leftrightarrow (U_{2,12}, U_{2,1}) \\ (U_{1,12}, U_{1,2}) &\leftrightarrow X_1 \leftrightarrow X_2 \leftrightarrow (U_{2,12}, U_{2,2}) \end{aligned} \tag{3}$$

where $X \leftrightarrow Y \leftrightarrow Z$ denotes that (X, Y, Z) form a Markov chain in that order. Note that $U_{i,\mathcal{S}}$ is sent in the packet from source i to sinks $j : j \in \mathcal{S}$. The encoding is divided into 3 stages.

Encoding : We first focus on the encoding at s_1. In the first stage, $2^{nR'_{1,12}}$ codewords of $U_{1,12}$, each of length n are generated independently, with elements drawn according to the marginal PMF $P(U_{1,12})$. Conditioned on each of these codewords, $2^{nR'_{1,1}}$ and $2^{nR'_{1,2}}$ codewords of $U_{1,1}$ and $U_{1,2}$ are generated according to the conditional PMFs $P(U_{1,1}|U_{1,12})$ and $P(U_{1,2}|U_{1,12})$ respectively. Codebooks for $U_{2,12}, U_{2,1}$ and $U_{2,2}$ are generated at s_2 in a similar fashion. On observing a sequence X_1^n, s_1 first tries to find a codeword tuple from the codebooks of $(U_{1,12}, U_{1,1}, U_{1,2})$ which are jointly typical with X_1^n. The probability of finding such a codeword tuple approaches 1 if $R'_{1,12} \geq I(X_1; U_{1,12})$, $R'_{1,1} \geq I(X_1; U_{1,1}|U_{1,12})$ and $R'_{1,2} \geq I(X_1; U_{1,2}|U_{1,12})$. Let the codewords selected be denoted by $(u_{1,12}, u_{1,1}, u_{1,2})$. Similar constraints on $(R'_{2,1}, R'_{2,2}, R'_{2,12})$ must be satisfied for encoding at s_2. Denote the codewords selected at s_2 by $(u_{2,12}, u_{2,1}, u_{2,2})$. It follows from (3) and the 'Conditional Markov Lemma' in [14] that $(x_1^n, x_2^n, u_{1,12}, u_{1,1}, u_{2,12}, u_{2,1}) \in \mathcal{T}_\epsilon^n$ and $(x_1^n, x_2^n, u_{1,12}, u_{1,2}, u_{2,12}, u_{2,2}) \in \mathcal{T}_\epsilon^n$, where $\mathcal{T}_\epsilon^n$ denotes the ϵ-typical set of length n sequences.

In the second stage of encoding, each encoder uniformly divides the $2^{nR'_{i,\mathcal{S}}}$ codewords of $U_{i,\mathcal{S}}$ into $2^{nR''_{i,\mathcal{S}}}$ bins $\forall\, i \in \{1,2\}$, $\mathcal{S} \in \{1, 2, 12\}$. All the codewords which have the same bin index m are said to fall in the bin $\mathcal{C}_{i,\mathcal{S}}(m)$ $\forall m \in (1 \ldots 2^{nR''_{i,\mathcal{S}}})$. Note that the number of codewords in bin $\mathcal{C}_{i,\mathcal{S}}(m)$ is $2^{n(R'_{i,\mathcal{S}} - R''_{i,\mathcal{S}})}$. If s_1 selects the codewords $(u_{1,12}, u_{1,1}, u_{1,2})$ in the first stage and if the bin indices associated with $(u_{1,12}, u_{1,1}, u_{1,2})$ are $(m_{1,12}, m_{1,1}, m_{1,2})$, then $m_{1,1}$ is routed to sink S_1, $m_{1,2}$ to sink S_2 and $m_{1,12}$ to both the sinks S_1 and S_2. Similarly, bin indices $(m_{2,12}, m_{2,1}, m_{2,2})$ are routed from s_2 to the corresponding sinks.

The third stage of encoding, resembles the 'Power Binning' scheme of [10]. Every typical sequence of X_1^n is assigned a random bin index uniformly chosen over $[1 : 2^{n\tilde{R}_{1,1}}]$. All sequences with the same index, l_1, form a bin $\mathcal{B}_{1,1}(l_{1,1})$ $\forall l_{1,1} \in \{1 \ldots 2^{n\tilde{R}_{1,1}}\}$. Upon observing a sequence $X_1^n \in \mathcal{T}_\epsilon^n$ with bin index $l_{1,1}$, in addition to $m_{1,1}$, s_1 also routes index $l_{1,1}$ to S_1. Similarly bin index $l_{2,2}$ is routed from s_2 to S_2 in addition to $m_{2,2}$. These bin indices are used to reconstruct X_1^n and X_2^n losslessly at the decoders. Note that, if source i is to be reconstructed at a subset of sinks $\Pi_i \subseteq \Pi$, the source assigns $2^{|\Pi_i|} - 1$ independently generated indices to each sequence and each index is routed to a subset of Π_i.

Decoding : We again focus on sink S_1. S_1 receives the indices $(m_{1,12}, m_{1,1}, m_{2,12}, m_{2,1}, l_{1,1}, l_{1,12}, l_{2,1}, l_{2,12})$. It first looks for a pair of unique codewords from $\mathcal{C}_{1,12}(m_{1,12})$ and $\mathcal{C}_{2,12}(m_{2,12})$ which are jointly typical. Obviously, there is at least one pair, $(u_{1,12}, u_{2,12})$, which is jointly typical. It can be easily shown using standard typicality arguments that, the probability that no other pair of codewords are jointly typical approaches 1 if:

$$\begin{aligned} R''_{1,12} &\geq I(X_1; U_{1,12}|U_{2,12}) \\ R''_{2,12} &\geq I(X_2; U_{2,12}|U_{1,12}) \\ R''_{1,12} + R''_{2,12} &\geq I(X_1, X_2; U_{1,12}, U_{2,12}) \end{aligned} \tag{4}$$

The decoder at S_1 next looks at the codebooks of $U_{1,1}$ and $U_{2,1}$ which were generated conditioned on $u_{1,12}$ and $u_{2,12}$ respectively to find a unique pair of codewords from $\mathcal{C}_{1,1}(m_{1,1})$ and $\mathcal{C}_{2,1}(m_{2,1})$ which are jointly typical with $(u_{1,12}, u_{2,12})$. We again have one pair, $(u_{1,1}, u_{2,1})$, which is jointly typical with $(u_{1,12}, u_{2,12})$. It can be shown using arguments similar to [8] that, the probability of finding no other jointly typical pair approaches 1 if :

$$\begin{aligned} (R'_{1,1} - R''_{1,1}) \leq -H(U_{1,1}|U_{2,1}, U_{1,12}, U_{2,12}) \\ +H(U_{1,1}|U_{1,12}) \end{aligned} \tag{5}$$

$$\begin{aligned} (R'_{2,1} - R''_{2,1}) \leq -H(U_{2,1}|U_{1,1}, U_{1,12}, U_{2,12}) \\ +H(U_{2,1}|U_{2,12}) \end{aligned} \tag{6}$$

$$\begin{aligned} \Big\{(R'_{1,1} - R''_{1,1}) + \quad &\leq \quad H(U_{1,1}|U_{1,12}) + H(U_{2,1}|U_{2,12}) \\ (R'_{2,1} - R''_{2,1})\Big\} \quad &\quad -H(U_{1,1}, U_{2,1}|U_{1,12}, U_{2,12}) \end{aligned} \tag{7}$$

On substituting the constraints $R^{'}_{1,1} \geq I(X_1;U_{1,1}|U_{1,12})$ and $R^{'}_{2,1} \geq I(X_2;U_{2,1}|U_{2,12})$, and using the Markov chain condition in (3) we get:

$$\begin{aligned} R^{''}_{1,1} &\geq I(X_1;U_{1,1}|U_{1,12},U_{2,12},U_{2,1}) \\ R^{''}_{2,1} &\geq I(X_2;U_{2,1}|U_{2,12},U_{1,12},U_{1,1}) \\ R^{''}_{1,1}+R^{''}_{2,1} &\geq I(X_1,X_2;U_{1,1},U_{2,1}|U_{1,12},U_{2,12}) \end{aligned} \quad (8)$$

After successfully decoding the codewords $(u_{1,12},u_{1,1},u_{2,12},u_{2,1})$, the decoder at S_1 looks for a unique sequence from $\mathcal{B}_{1,1}(l_{1,1})$ which is jointly typical with $(u_{1,12},u_{1,1},u_{2,12},u_{2,1})$. We again have x_1^n satisfying this property. It can be shown that the probability of finding no other sequence which is jointly typical with $(u_{1,12},u_{1,1},u_{2,12},u_{2,1})$ approaches 1 if:

$$\tilde{R}_{1,1} \geq H(X_1|U_{1,12},U_{1,1},U_{2,12},U_{2,1}) \quad (9)$$

Similar constraints on rates can be obtained for lossless decoding at S_2. The first packet from s_1, destined to only S_1, carries indices $(m_{1,1},l_{1,1})$ at rate $R_{1,1}=R^{''}_{1,1}+\tilde{R}_{1,1}$. The second and third packets carry $m_{1,2}$ and $m_{1,12}$ at rates $R^{''}_{1,2}$ and $R^{''}_{1,12}$ respectively and are routed to the corresponding sinks. Similarly, 3 packets are transmitted from s_2 carrying indices $\{m_{2,1},m_{2,12},(m_{2,2},l_{2,2})\}$ at rates $(R^{''}_{2,1},R^{''}_{2,12},R^{''}_{2,2}+\tilde{R}_{2,2})$ to sinks $\{S_1,S_2,(S_1,S_2)\}$ respectively. Achievable rates for $(R_{1,1},R_{1,2},R_{1,12},R_{2,1},R_{2,2},R_{2,12})$ can now be obtained using (4), (8) and (9). The convex closure of achievable rates over all such random variables $(U_{1,12},U_{1,1},U_{1,2},U_{2,12},U_{2,1},U_{2,2})$ gives the achievable rate region for the 2 source - 2 sinks DIR problem.

III. Dispersive Information Routing with Helpers - General Setup

Let a network be represented by an undirected connected graph $G=(V,E)$. Each edge $e\in E$ is associated with an edge weight, w_e. The communication cost is assumed to be a simple product of the edge rate and edge weight, i.e. $C_e=r_e w_e$[2]. The nodes V consist of N source nodes (denote by $s_1,s_2\dots s_N$), M sinks (denote by $S_1,S_2\dots S_N$), and $|V|-N-M$ intermediate nodes. We denote by $\Sigma=\{1\dots N\}$ and $\Pi=\{1\dots M\}$. Source node s_i observes random variable X_i distributed over a finite alphabet $\mathcal{X}_i$. Sink S_j reconstructs (requests) a subset of the sources denoted by $\Sigma_j\subseteq\Sigma$. Conversely, source s_i is reconstructed at a subset of sinks denoted by $\Pi_i\subseteq\Pi$. The objective is to find the minimum communication cost achievable by dispersive information routing for lossless reconstruction of the requested sources at each sink when every source can (possibly) communicate with every sink.

In what follows, $2^{\mathcal{S}}$ denotes the set of all subsets (power set) of any set $\mathcal{S}$ and $|\mathcal{S}|$ denotes the set cardinality. Note that $|2^{\mathcal{S}}|=2^{|\mathcal{S}|}$. $\mathcal{S}^c$ denotes the set complement and ϕ denotes the null set. For two sets $\mathcal{S}_1$ and $\mathcal{S}_2$, we denote the set difference by $\mathcal{S}_1-\mathcal{S}_2=\{\mathcal{K}:\mathcal{K}\in\mathcal{S}_1,\mathcal{K}\notin\mathcal{S}_2\}$. We denote by $2^{\mathcal{S}}-\phi$, the set of all non-empty subsets of $\mathcal{S}$. We use the shorthand $\{U_i\}_{\mathcal{S}}$ for $\{U_{i,\mathcal{K}}:\mathcal{K}\in\mathcal{S}\}$ and $\{U_\Gamma\}_{\mathcal{S}}$ for $\{U_{i,\mathcal{K}}:i\in\Gamma,\mathcal{K}\in\mathcal{S}\}$[3].

A. Obtaining the effective costs

Under DIR each source transmits 2^M-1 packets into the network, each meant for a different subset of sinks. Let the packet from source s_i to the subset of sinks $\mathcal{S}\in 2^{\Pi}$ be denoted by $\mathcal{P}_{i,\mathcal{S}}$ and let it carry information at rate $R_{i,\mathcal{S}}$. The optimum route for packet $\mathcal{P}_{i,\mathcal{S}}$ from the source to these sinks is determined by a spanning tree optimization (minimum Steiner tree) [13]. The minimum cost of transmitting packet $\mathcal{P}_{i,\mathcal{S}}$ with $R_{i,\mathcal{S}}$ bits from source i to the subset of sinks $\mathcal{S}$, denoted by $d_i(\mathcal{S})$ is :

$$d_i(s)=R_{i,\mathcal{S}}\times\min_{Q\in E_{i,S}}\sum_{e\in Q}w_e \quad (10)$$

where $E_{i,\mathcal{S}}$ denotes the set of all paths from source i to the subset of sinks $\mathcal{S}$. Our next objective is to find an achievable rate region for the tuple $(R_{i,\mathcal{S}}\ \forall i\in\Sigma,\mathcal{S}\in 2^{\Pi})$. The minimum communication cost then follows directly from a simple linear programming formulation.

B. An achievable rate region

We extend the coding scheme described in section II to derive an achievable rate region for the tuple $(R_{i,\mathcal{S}}\ \forall i\in\Sigma,\mathcal{S}\in 2^{\Pi}-\phi)$ using principles from Multiple Descriptions encoding [11], albeit with more complex notation. Without loss of generality, we assume that, every source can send packets to every sink. The corresponding costs can be set to ∞ if some paths do not exist.

Before stating the achievable rate region in Theorem 1, we define the following subsets of 2^{Π}:

$$\begin{aligned} \mathcal{I}_W &= \{\mathcal{S}:\mathcal{S}\in 2^{\Pi},\ |\mathcal{S}|=W\} \\ \mathcal{I}_{W+} &= \{\mathcal{S}:\mathcal{S}\in 2^{\Pi},\ |\mathcal{S}|>W\} \end{aligned}$$

Let $\mathcal{B}$ be any subset of Π with $|\mathcal{B}|\leq W$. We define the following subsets of $\mathcal{I}_W$ and $\mathcal{I}_{W+}$:

$$\begin{aligned} \mathcal{I}_W(\mathcal{B}) &= \{\mathcal{S}:\mathcal{S}\in\mathcal{I}_W,\ \mathcal{B}\subseteq\mathcal{S}\} \\ \mathcal{I}_{W+}(\mathcal{B}) &= \{\mathcal{S}:\mathcal{S}\in\mathcal{I}_{W+},\ \mathcal{B}\subseteq\mathcal{S}\} \end{aligned}$$

We also define $\mathcal{J}(\mathcal{K})=\{\mathcal{S}:\ \mathcal{S}\in 2^{\Pi},\ |\mathcal{K}\bigcap\mathcal{S}|>0\}$

Let $\{U_\Sigma\}_{2^{\Pi}-\phi}$ be any set of $N(2^M-1)$ random variables defined on arbitrary finite sets, jointly distributed with $\{X\}_\Sigma$, such that $\forall j\in\Pi$:

$$P(\{X\}_\Sigma,\{U_\Sigma\}_{\mathcal{J}(j)})=P(\{X\}_\Sigma)\prod_{i\in\Sigma}P(\{U_i\}_{\mathcal{J}(j)}|X_i) \quad (11)$$

The above Markov condition ensures that all the codewords which reach a sink are jointly typical with $\{X\}_{\Sigma_j}$. We define $\beta_j(W,\mathcal{Q}_1,\dots,\mathcal{Q}_N)\ \forall j\in\Pi, W\in\Pi, \mathcal{Q}_i\subseteq\mathcal{I}_W(j)$ as:

$$\begin{aligned} \beta_j(W,\mathcal{Q}_i\forall i) &= -\sum_{i\in\Sigma}H\left(\{U_i\}_{\mathcal{Q}_i}|\{U_i\}_{\mathcal{I}_{W+}},X_i\right) \\ &\quad + H\left(\{U_i\}_{\mathcal{Q}_i}\forall i|\{U_i\}_{\mathcal{Q}_i^c},\{U_i\}_{\mathcal{I}_{W+}(j)}\forall i\right) \end{aligned} \quad (12)$$

[2]The approach is applicable to more general cost functions.

[3]Note the difference between $\{U_i\}_{\mathcal{S}}$ and $U_{i,\mathcal{S}}$. $\{U_i\}_{\mathcal{S}}$ is a set of variables, whereas $U_{i,\mathcal{S}}$ is a single variable.

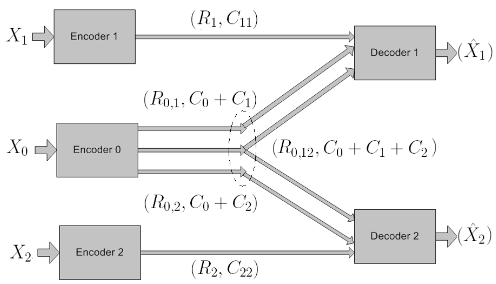


Figure 2: Example of a 2-sink, 1 Helper DIR

where $\mathcal{Q}_i^c = \mathcal{I}_W - \mathcal{Q}_i$. We also define $\gamma_j(\Gamma)\ \forall j \in \Pi, \Gamma \subseteq \Sigma_j$ as:

$$\gamma_j(\Gamma) = H\left(\{X\}_\Gamma | \{X\}_{\Gamma^c}, \{U_\Sigma\}_{\mathcal{I}_{W+}(j)}\right) \tag{13}$$

where $\Gamma^c = \Sigma_j - \Gamma$. We state our main result in the following theorem.

Theorem 1. *Let $\{U_\Sigma\}_{2^\Pi - \phi}$ be any set of random variables satisfying (11). Let $R''_{i,\mathcal{S}}\ \forall i \in \Sigma, \mathcal{S} \in 2^\Pi - \phi$ be any set of rate tuples* such that, $\forall j \in \Pi, W \in \{1, \ldots, M\}, \mathcal{Q}_i \subseteq \mathcal{I}_W(j)$*:*

$$\sum_{i \in \Sigma} \sum_{\mathcal{S} \in \mathcal{Q}_i} R''_{i,\mathcal{S}} \quad \geq \quad \max(\beta_j(W, \mathcal{Q}_1, \ldots, \mathcal{Q}_N), 0) \tag{14}$$

and let $\tilde{R}_{i,\mathcal{S}}\ \forall i \in \Sigma, \mathcal{S} \in 2^{\Pi_j} - \phi$ satisfy:

$$\sum_{i \in \Gamma} \sum_{\mathcal{S}: j \in \mathcal{S}} \tilde{R}_{i,\mathcal{S}} \geq \gamma_j(\Gamma) \tag{15}$$

$\forall j \in \Pi, \Gamma \in 2^{\Sigma_j} - \phi$. Then, the achievable rate region for the tuple $(R_{i,\mathcal{S}}\ \forall i \in \Sigma, \mathcal{S} \in 2^\Pi - \phi)$ contains all rates such that,

$$R_{i,\mathcal{S}} \geq \begin{cases} R''_{i,\mathcal{S}} + \tilde{R}_{i,\mathcal{S}} & \text{if } \mathcal{S} \subseteq 2^{\Pi_i} - \phi \\ R''_{i,\mathcal{S}} & \text{if } \mathcal{S} \nsubseteq 2^{\Pi_i} - \phi \end{cases} \tag{16}$$

The convex closure of the achievable tuples over all such $N(2^M - 1)$ random variables satisfying (11) is the achievable rate region for DIR and is denoted by $\mathcal{R}_{DIR}$.

Proof: Omitted due to space constraints. ∎

We note that the converse to this achievability region does not hold in general. However, we can prove that the converse holds for the following two important non-trivial special cases:

(1) *When there are no helpers* : When there are no helpers, setting $U_{i,\mathcal{S}} = \Phi\ \forall i \in \Sigma, \mathcal{S} \in 2^\Pi - \phi$, where Φ is a constant, leads to the rate region in [10]. The converse to this rate region follows directly from arguments similar to the converse of Slepian-Wolf Theorem [5].

(2) *A 2 sink network with a single helper* : The converse can be proven in general for *any* 2 sink network with a single helper. We omit the details here due to space constraints. However, we just give a simple example of a 2 sink network with a single helper. Consider the network shown in Figure 2, with 3 sources and 2 sinks. Note that s_0 acts as a helper to both the sinks. The rate region of Theorem 1 for the tuple $(R_{1,1}, R_{2,2}, R_{0,12}, R_{0,1}, R_{0,2})$ simplifies to the following. The set of achievable rate tuple for each (U_0, U_1, U_2) jointly distributed with (X_1, X_0, X_2) such that $X_1 \leftrightarrow X_0 \leftrightarrow (U_0, U_1, U_2)$ and $X_2 \leftrightarrow X_0 \leftrightarrow (U_0, U_1, U_2)$ is given by $R_{0,12} \geq I(X_0; U_0)$, $R_{0,1} \geq I(X_0; U_1|U_0)$, $R_{0,2} \geq I(X_0; U_2|U_0)$, $R_{1,1} \geq H(X_1|U_0, U_1)$ and $R_{2,2} \geq H(X_2|U_0, U_2)$. The closure over all such (U_0, U_1, U_2) is the complete rate region for this problem. The converse follows in similar lines to the derivation of the outer bound in [8].

IV. Conclusion

This paper considers a new routing paradigm called dispersive information routing, wherein each intermediate node is allowed to split a packet and forward subsets of the packet on each forward path. In our prior work, we considered a special case of the problem when each sink is allowed to receive packets only from the sources it intends to reconstruct, and derived the complete rate region. In this paper, considered a more general framework wherein each sink can (possibly) receive packets from all the sources. Unfortunately, the problem becomes considerably more complex. We derived an achievable rate region using principles from multiple descriptions encoding and Han and Kobayashi decoding which is complete only for certain special cases of the setup.

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