Zero-error source-channel coding with source side information at the decoder

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Let $\mathcal{C}$ be a discrete memoryless channel with transition probability distribution $p_{Y|X}(y|x), x \in \mathcal{X}, y \in \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are finite sets. Let $(\mathcal{S}_U, \mathcal{S}_V)$ be a pair of memoryless correlated sources producing a pair of random variables $(U,V) \sim p_{UV}(u,v)$ from a finite set $\mathcal{U} \times \mathcal{V}$ at each instant. Alice, the sender, has access to U while Bob, the receiver, has access to V. Alice and Bob are connected by the channel $\mathcal{C}$. We wish to decide if Alice can convey n realizations of U in n channel uses for some n. The asymptotically vanishing probability of error case offers the following possibility: By Slepian-Wolf, we can encode U at the rate $H(U|V)$. Channel coding makes reliable transmission possible if $H(U|V)$ is less than the Shannon capacity of the channel. Shamai and Verdu [1] showed that this strategy of separate source and channel coding cannot be improved upon. The situation changes dramatically if we impose a zero-error constraint. *Our main results are that separate coding is asymptotically suboptimal and that in fact the gains by joint source-channel coding can be unbounded.*

To analyze the zero-error scenario, we define the following graphs: the source confusability graph $G_U = (\mathcal{U}, E_U)$ where $(u_1, u_2) \in E_U (\subseteq \mathcal{U} \times \mathcal{U})$ iff $\exists v \in \mathcal{V} : p_{UV}(u_1, v) p_{UV}(u_2, v) > 0$; the channel characteristic graph $G_X = (\mathcal{X}, E_X)$ where $(x_1, x_2) \in E_X (\subseteq \mathcal{X} \times \mathcal{X})$ iff $\exists y \in \mathcal{Y} : p_{Y|X}(y|x_1) p_{Y|X}(y|x_2) > 0$. A scalar source-channel code is a mapping $f : \mathcal{U} \to \mathcal{X}$ such that source symbols that are not distinguishable on the basis of the side information, must have distinguishable images under f. In terms of the graphs defined above, the condition on the map f is: $(u_1, u_2) \in E_U \Rightarrow (f(u_1) \neq f(u_2))$ and $(f(u_1), f(u_2)) \notin E_X$. Note that we can always find a code from a source graph to its complement. Separate source and channel coding corresponds to first mapping from the source alphabet to an index set and then mapping from the index set to the channel alphabet. The size of the smallest index set that allows a zero-error source code is the chromatic number of G_U, $\chi(G_U)$. The largest index set from which we can map to the channel alphabet is the stability number of G_X, $\alpha(G_X)$. We consider two coding scenarios for block length n depending on the extension of G_U used (for motivation, see [2]):

1. *Unrestricted Input (UI)*: The relevant source graph is the n-fold OR product of G_U, $G_U^{(n)}$.
2. *Restricted Input (RI)*: The relevant source graph is the n-fold AND product of G_U, G_U^n.

The extension of the channel graph to be considered in both cases is its n-fold AND product, G_X^n. The asymptotic minimum source coding rates are the fractional chromatic number $R^*(G_U) = \lim_{n\to\infty} \frac{1}{n} \log \chi(G_U^{(n)})$ bits per symbol in the UI case and the Witsenhausen rate, $R_w(G_U) = \lim_{n\to\infty} \frac{1}{n} \log \chi(G_U^n)$ bits per symbol in the RI case. The maximum allowable rate for the channel is the capacity of the characteristic graph, $C(G_X) = \lim_{n\to\infty} \frac{1}{n} \log \alpha(G_X^n)$ bits per channel use. Observe that all these rates are purely functions of the associated graphs.

Unrestricted Input case: Our first result is that separate coding is suboptimal in the UI case. Source and channel coding can be done separately only if $R^*(G_U) \leq C(G_X)$. Consider the pentagon graph $G = (\{0,1,2,3,4\}, E)$, where $(i,j) \in E \Leftrightarrow |i-j| = 1(\text{mod}5)$. Let $G_U = G_X = G$. $R^*(G_U) = \log \frac{5}{2} > C(G_X) = \frac{1}{2}\log 5$. So, we cannot transmit the source through the channel by separate coding. On the other hand, a scalar joint source-channel code exists since the pentagon graph is its own complement. Therefore, separate source and channel coding is suboptimal.

Restricted Input case: In this case separate source and channel coding is possible only if $R_w(G_U) \leq C(G_X)$. Our proof of the suboptimality of separate coding employs the theta function, $\vartheta(G)$, defined by Lovász. Lovász proved that $\log \vartheta(G) \geq C(G)$ for any graph G. One of our key results is:

Theorem 1. For any graph $G = (V,E)$, $\log \vartheta(\bar{G}) \leq R_w(G)$.

We now have the string of inequalities $C(G) \leq \log \vartheta(G) \leq R_w(\bar{G})$. If any of these inequalities is strict for some graph, then separate coding is suboptimal in the RI case as well. Indeed, such graphs do exist. For example, if G_X is the Schläfli graph, $C(G_X) \leq \log 7 < \vartheta(G_X) = \log 9$.

How large are the gains? Given a source-channel pair (G_U, G_X), let us rephrase the problem as: how many channel uses are required per source symbol to enable zero-error transmission? With separate coding, the channel uses per symbol in the UI case is $\frac{R^*(G_U)}{C(G_X)}$ while in the RI case it is $\frac{R_w(G_U)}{C(G_X)}$. Using a recent result by Alon [3], we show that:

Theorem 2. Given any l, we can find a graph G such that

$$\frac{R^*(\bar{G})}{C(G)} \geq \frac{R_w(\bar{G})}{C(G)} \geq l.$$

This means that both $\frac{R^*(G_U)}{C(G_X)}$ and $\frac{R_w(G_U)}{C(G_X)}$ can be arbitrarily large even when $G_U = \overline{G_X}$, the case where a zero-error (scalar) joint source-channel code exists with one channel use per source symbol.

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