Additive Successive Refinement

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Scalable coding rate-distortion bounds, and conditions for their coincidence with non-scalable rate-distortion bounds, were derived in [4, 3]. However, there has always been an implicit assumption of *tree structured vector quantization* (TSVQ) scheme, which requires extensive memory usage. In most of the practical applications, on the other hand, *additive* structures such as *multi-stage vector quantization* (MSVQ) are preferred. In this paper, we analyze the bounds for MSVQ.

Let $\{X_t\}_{t=1}^{\infty}$ be a discrete memoryless source taking values from an alphabet $\mathcal{X}$. We encode this source in two MSVQ stages, i.e., we have two stage-encoding functions $f_i : \mathcal{X}^n \to \mathcal{F}_i$, and stage-decoding functions $g_i : \mathcal{F}_i \to \hat{\mathcal{X}}^n$, where $\hat{\mathcal{X}}$ is the reproduction alphabet. We adopt the notation $\hat{X}_1^n = g_1(f_1(X^n))$ and $\hat{X}_2^n = g_2(f_2(X^n))$ for stage decoder outputs, and $\hat{Y}_1^n = \hat{X}_1^n$ and $\hat{Y}_2^n = \hat{X}_1^n + \hat{X}_2^n$ for reconstructed vectors, where $(\hat{\mathcal{X}}, +)$ forms an Abelian (i.e., commutative) group, and $+$ is a "single-letter" operator. The quadruple (R_1, R_2, D_1, D_2) is called (α,β)-achievable if there exist stage-encoding functions f_i and stage-decoding functions g_i with

$$n(R_i + \alpha) \geq \log|\mathcal{F}_i| \quad i = 1, 2. \tag{1}$$

$$n(D_1 + \beta) \geq E\sum_{t=1}^{n} d(X_t, \hat{X}_{1t}) \tag{2}$$

$$n(D_2 + \beta) \geq E\sum_{t=1}^{n} d(X_t, \hat{X}_{1t} + \hat{X}_{2t})\,. \tag{3}$$

Theorem 1 (Sufficient Conditions for Achievability) *If there exist random variables $\hat{X}_1$ and $\hat{X}_2$, jointly distributed with source variable X, such that*

$$\begin{aligned} I(X;\hat{X}_1) &\leq R_1 + \alpha \\ I(X;\hat{X}_2) &\leq R_2 + \alpha \\ I(X;\hat{X}_1,\hat{X}_2) + I(\hat{X}_1;\hat{X}_2) &\leq R_1 + R_2 + 2\alpha \\ Ed(X,\hat{X}_1) &\leq D_1 + \beta \\ Ed(X,\hat{X}_1+\hat{X}_2) &\leq D_2 + \beta\,, \end{aligned} \tag{4}$$

then the quadruple (R_1, R_2, D_1, D_2) is (α,β)-achievable.

The proof is very similar to the one used by El Gamal and Cover for the multiple descriptions problem [2].

Theorem 2 (Necessary Conditions for Achievability) *If the quadruple (R_1, R_2, D_1, D_2) is (α,β)-achievable, then there exist random variables $\hat{X}_1$ and $\hat{X}_2$, jointly distributed with source variable X, such that*

$$\begin{aligned} I(X;\hat{X}_1) &\leq R_1 + \alpha \\ I(X;\hat{X}_2) &\leq R_2 + \alpha \\ I(X;\hat{X}_1,\hat{X}_2) &\leq R_1 + R_2 + 2\alpha \\ Ed(X,\hat{X}_1) &\leq D_1 + \beta \\ Ed(X,\hat{X}_1+\hat{X}_2) &\leq D_2 + \beta\,. \end{aligned} \tag{5}$$

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We call (R_1, R_2, D_1, D_2) *achievable*, if it is (α,β)-achievable for all $\alpha,\beta > 0$.

Definition 1 (Scalable Coding Bound) [4] *The rate-distortion bound for scalable coding is given by*

$$R(D_1, D_2, R_1) = \inf_{\substack{p(\hat{y}_1,\hat{y}_2|x) \\ \text{s.t. } I(X;\hat{Y}_1) \leq R_1 \\ Ed(X,\hat{Y}_1) \leq D_1 \\ Ed(X,\hat{Y}_2) \leq D_2}} I(X;\hat{Y}_1,\hat{Y}_2) \tag{6}$$

Theorem 3 (The Special Case of No-excess-rate) *When $R_1 + R_2 = R(D_1, D_2, R_1)$, a quadruple (R_1, R_2, D_1, D_2) is achievable if and only if there exist random variables $\hat{X}_1$ and $\hat{X}_2$, jointly distributed with source variable X, such that*

$$\begin{aligned} I(X;\hat{X}_1) &\leq R_1 \\ I(X;\hat{X}_2) &\leq R_2 \\ I(\hat{X}_1;\hat{X}_2) &= 0 \\ I(X;\hat{X}_1,\hat{X}_2) &\leq R_1 + R_2 \\ Ed(X,\hat{X}_1) &\leq D_1 \\ Ed(X,\hat{X}_1+\hat{X}_2) &\leq D_2\,. \end{aligned} \tag{7}$$

The proof follows virtually the same lines as the multiple descriptions proof in [1, Section IV].

Example: Let $\mathcal{X} = \hat{\mathcal{X}} = \mathcal{R}$, and let $X \sim p(x)$. If the Shannon Lower Bound (SLB) is tight at distortion D_1, we observe that $p(x,\hat{x}_1,\hat{x}_2)$ satisfies all conditions in (7) for $D_2 < D_1$ and $R_1 = R(D_1)$, $R_1 + R_2 = R(D_1, D_2, R_1) = R(D_2)$ if

$$p(x,\hat{x}_1,\hat{x}_2) = \frac{q(\hat{x}_1)e^{-\frac{1}{2D_2}(x-\hat{x}_1-\hat{x}_2)^2 - \frac{1}{2(D_1-D_2)}\hat{x}_2^2}}{2\pi\sqrt{D_2(D_1-D_2)}}\,,$$

where $q(\hat{x}_1)$ is the reproduction pdf that achieves D_1 and $R_1 = R(D_1)$. Hence, when SLB is tight, lossless successive refinement is possible not only in the TSVQ sense, but also in the additive-coding (MSVQ) sense.

Counterexample: Let $\mathcal{X} = \hat{\mathcal{X}} = \{0, 1, 2\}$, with Hamming distortion measure, and let "+" be defined as modulo 3 summation. Let $p_0 \geq p_1 \geq p_2$. This source is successively refinable in the usual (TSVQ) sense [3], i.e., $R(D_1, D_2, R_1) = R(D_2)$. However, the conditions (7) are not satisfied for any $D_1 > 2p_2$. Hence this is an example of a source that is everywhere successively refinable but is not everywhere additively refinable.

References

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