On Variable-Length Coding of Sources with Side Information at Multiple Decoders

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I. Introduction

Let the source sequence $X(\cdot)$ and the side-information sequences $Y_1(\cdot), \ldots, Y_k(\cdot)$ be discrete-time signals available at transmitter $\mathcal{T}$, and receivers $\mathcal{R}_1, \ldots, \mathcal{R}_k$, respectively. For each $t = 1, 2, \ldots$, let $X(t) \in \mathcal{X}$ and $Y_i(t) \in \mathcal{Y}_i$ for $i = 1, \ldots, k$, where $\mathcal{X}$ and $\mathcal{Y}_i$ are finite alphabets. Assume that the joint set of samples $[X(t), Y_1(t), \ldots, Y_k(t)]$ are temporally independent and identically distributed (i.i.d.). The task of the transmitter is to broadcast a single description for $X(\cdot)$ such that simultaneous *zero-error* reproduction of $X(\cdot)$ is possible at each receiver $\mathcal{R}_i$ (having access only to the side-information $Y_i(\cdot)$). We are interested in the asymptotically achievable minimum rate when the transmitter uses variable-length block coding.

The simplest case $k = 1$ has been extensively investigated by several researchers. Witsenhausen [4] observed that there is a one-to-one relationship with valid colorings of (the powers of) the *characteristic graph* and valid fixed-length (block) codes. Alon and Orlitsky [1] gave a characterization of the minimum rate for the general case of variable-length coding, in terms of the limit of the normalized *chromatic entropy* of the powers of the characteristic graph. Koulgi et al. [2] proved that the minimum rate given in [1] is in fact equal to the *complementary graph entropy* of the characteristic graph. The problem for general k, under fixed-length coding constraint, was recently introduced by Simonyi [3]. He defined a characteristic graph family for the problem and proved that the minimum rate needed for zero-error transmission is given by the maximum Witsenhausen rate over all the graphs in the family. In this paper, we generalize Simonyi's result to the case of variable-length coding. Our result shows that the asymptotic minimum rate is given by the maximum complementary graph entropy in the characteristic graph family.

II. Results

Define the characteristic graph family $\mathcal{G} = \{G_1, \ldots, G_k\}$ as follows. The vertex set $V(G_i) = \mathcal{X}$ is the same for all graphs. Two elements, a and b of $\mathcal{X}$, form an edge in G_i if and only if there exists $c \in \mathcal{Y}_i$ jointly possible with both a and b, i.e., $\{a, b\} \in E(G_i)$ if and only if $\exists c : P_{X,Y_i}(a, c)P_{X,Y_i}(b, c) > 0$. We use the notation G^n for the n-fold *AND product* of G with itself. That is, two sequences of length n are connected in G^n if and only if they are adjacent in G at every coordinate where they are not equal. For $\epsilon > 0$, let $G_i^n(P_X, \epsilon)$ denote the graph *induced* in G_i^n by the (P_X, ϵ)-strongly-typical sequences in $\mathcal{X}^n$. Finally, let the union $\cup_i G_i$ be defined by $V(\cup_i G_i) = \mathcal{X}$ and $E(\cup_i G_i) = \cup_i E(G_i)$.

It was shown in [3] that valid fixed-length block codes correspond to colorings of $\cup_i G_i^n$. Thus, the asymptotic minimum fixed-length rate is given by

$$R^{\mathrm{fl}}(\mathcal{G}) \triangleq \lim_{n\to\infty} \frac{1}{n} R_n^{\mathrm{fl}}(\mathcal{G}) = \lim_{n\to\infty} \frac{1}{n} \log_2 \chi(\cup_i G_i^n) ,$$

where $\chi(\cdot)$ denotes the *chromatic number* of a graph. Similarly, a variable-length block code is *instantaneously decodable* (or simply instantaneous) if and only if it assigns *prefix-free* codewords to adjacent vertices in $\cup_i G_i^n$. Therefore a valid coloring of $\cup_i G_i^n$ followed by Huffman coding of the colors constitutes an instantaneous code. Even though this approach may lead to suboptimal codes for finite n, we prove that it is asymptotically optimal:

$$R^{\mathrm{vl}}(\mathcal{G}, P_X) \triangleq \lim_{n\to\infty} \frac{1}{n} R_n^{\mathrm{vl}}(\mathcal{G}, P_X) = \lim_{n\to\infty} \frac{1}{n} H_\chi(\cup_i G_i^n, P_X^n) ,$$

where $H_\chi(\cdot,\cdot)$ denotes the *chromatic entropy* defined in [1]. It was shown in [2] that for $k = 1$, an alternative formula for the minimum variable-length coding rate is given by the complementary graph entropy of the single graph G. We begin by generalizing that result to $k > 1$.

Theorem 1:

$$\lim_{n\to\infty} \frac{1}{n} H_\chi(\cup_i G_i^n, P_X^n) = \overline{H}(\mathcal{G}, P_X) ,$$

where

$$\overline{H}(\mathcal{G}, P_X) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log_2 \chi(\cup_i G_i^n(P_X, \epsilon))$$

is the complementary graph entropy of the family $\mathcal{G}$. ◇

The proof generally follows similar lines to the proof of Theorem 1 in [2].

Since a valid variable-length code for the family of graphs $\mathcal{G}$ is also valid for each graph $G_i \in \mathcal{G}$, it follows that

$$\overline{H}(\mathcal{G}, P_X) = R^{\mathrm{vl}}(\mathcal{G}, P_X) \geq \max_i R^{\mathrm{vl}}(G_i, P_X) = \max_i \overline{H}(G_i, P_X) .$$

Simonyi [3] showed the fixed-length version of this trivial estimation, namely, $R^{\mathrm{fl}}(\mathcal{G}) \geq \max_i R^{\mathrm{fl}}(G_i)$, is actually always satisfied with equality. We generalize this result to variable-length coding, and prove the following theorem.

Theorem 2:

$$R^{\mathrm{vl}}(\mathcal{G}, P_X) = \max_i R^{\mathrm{vl}}(G_i, P_X) .$$

◇

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