

# Additive Successive Refinement

Ertem Tuncel, *Student Member, IEEE*, and Kenneth Rose, *Fellow, IEEE*

**Abstract**—Rate-distortion bounds for scalable coding, and conditions under which they coincide with non-scalable bounds, have been extensively studied. These bounds have been derived for the general tree-structured refinement scheme, where reproduction at each layer is an arbitrarily complex function of all encoding indexes up to that layer. However, in most practical applications (e.g., speech coding) “additive” refinement structures such as the multistage vector quantizer are preferred due to memory limitations. We derive an achievable region for the additive successive refinement problem, and show via a converse result that the rate-distortion bound of additive refinement is above that of tree-structured refinement. Necessary and sufficient conditions for the two bounds to coincide are derived. These results easily extend to abstract alphabet sources under the condition  $E\{d(X, a)\} < \infty$  for some letter  $a$ . For the special cases of square-error and absolute-error distortion measures, and subcritical distortion (where the Shannon lower bound (SLB) is tight), we show that successive refinement without rate loss is possible not only in the tree-structured sense, but also in the additive-coding sense. We also provide examples which are successively refinable without rate loss for all distortion values, but the optimal refinement is not additive.

**Index Terms**—Additive refinement, multistage vector quantization (MSVQ), rate distortion, scalable source coding, successive refinement.

## I. INTRODUCTION

**S**UCCESSIVE refinement of information, or scalable coding, refers to hierarchical coding of the source, where coarser descriptions are embedded in finer ones. In heterogeneous networks such as the Internet, where users may have different bandwidth constraints, this coding scheme is very appealing because of its capability to serve a diverse set of users. While users with high-bandwidth connections can achieve a high-quality reproduction of the source by accessing the entire coded bitstream, users with low-bandwidth constraints can also utilize the same bitstream by accessing only the embedded coarse descriptions and achieve a low-quality reproduction. In certain applications, such as image coding for progressive transmission, users can stop the communication process once they are satisfied with the accuracy level.

From the viewpoint of rate-distortion theory, the problem of scalable coding was initially addressed in the context of *successive*

*refinement without rate loss* by Koshelev [10], [11], and by Equitz and Cover [6]. A sufficient condition was given in [10], [11] for simultaneous achievability of the rate-distortion bound in all layers. In [6], the authors proved that the same condition is also necessary. Rimoldi [12] addressed the achievability of any sequence of rates and distortions and provided a complete characterization of the achievable region. Effros [5] extended these results for stationary ergodic and nonergodic sources.

In all prior work, it was implicitly assumed that scalability is accomplished by using a tree-structured decoding scheme, i.e., reproduction at layer  $L$  is an unconstrained function of all encoding indices from layer 1 to layer  $L$ . Of course, tree structured decoding is not a special case, but rather, it is the most general, and hence the optimal scalable coding strategy. However, its natural implementation, tree-structured vector quantization (TSVQ) [9], is usually impractical due to its high codebook storage requirements and demand for a huge training set in the design stage. To mitigate these complexity barriers, in most practical applications such as speech coding, a special case of TSVQ, namely, the multistage vector quantization (MSVQ) [9], is preferred. MSVQ is what we call an “additive” refinement structure, because refinement is based on adding a new vector, which is a function of only the current layer encoding index, to the previous layer reconstruction. In Fig. 1, the decoding structures of TSVQ and MSVQ are compared.

In this paper, motivated by the popularity of MSVQ, we investigate the rate-distortion performance of additive successive refinement coders with two layers. We first derive an achievable region for the quadruples  $(R_1, R_2, D_1, D_2)$ , i.e., the distortion and rates at each layer. A straightforward converse result, of course, is that no additively refined coder can outperform the general scalable coding bound of [12]. We improve this result by deriving a tighter converse region. We then focus on whether, and under what conditions, additive refinement is optimal, i.e., achieves the scalable coding bound. Observing the similarity of this requirement to the no-excess-rate case of the multiple descriptions problem [7], [1], we borrow a technique introduced in [1] to derive the necessary and sufficient conditions for the optimality of additive refinement.

All the results established in this work easily extend to abstract alphabet sources under the usual regularity condition, i.e., the existence of a reference letter  $a$  such that  $E\{d(X, a)\} < \infty$ . Based on this extension, we show that additive refinement is optimal for any continuous source with tight Shannon lower bound (SLB) [2], under square-error and absolute-error distortion measures. (The simplest examples in this respect are the Gaussian and the Laplacian sources, respectively, where SLB is tight for all distortion values of interest.) It is known [6] that for both the square-error and absolute-error distortion measures, when SLB is tight for  $D_1$  and  $D_2$ , successive refinement is possible without

Manuscript received August 17, 2001; revised March 26, 2003. This work was supported in part by the National Science Foundation under Grants EIA-9986057 and EIA-0080134, the University of California MICRO Program, Dolby Laboratories, Inc., Lucent Technologies, Inc., Mindspeed Technologies, Inc., and Qualcomm, Inc. The material in this paper was presented in part at the International Symposium on Information Theory, Washington DC, June 2001.

The authors are with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, Santa Barbara, CA 93106-9560 USA (e-mail: ertem@ece.ucsb.edu; rose@ece.ucsb.edu).

Communicated by I. E. Telatar, Associate Editor for Shannon Theory.

Digital Object Identifier 10.1109/TIT.2003.814934

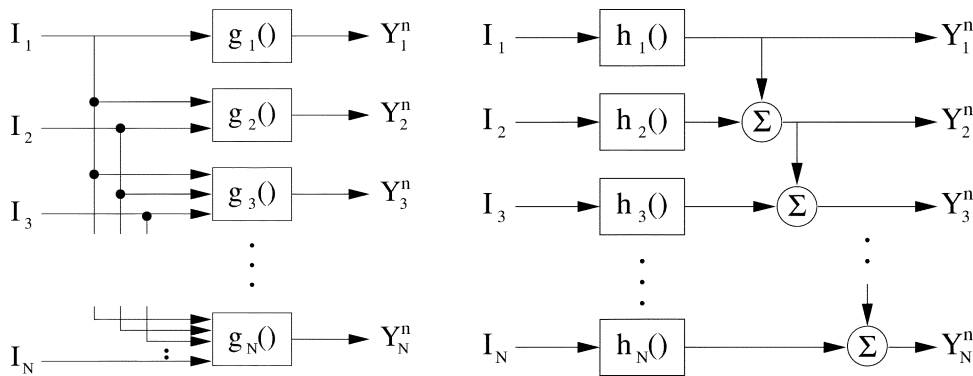


Fig. 1. The TSVQ (left) and MSVQ (right) decoders.

any rate loss. Our result shows further that these circumstances also guarantee *additive* successive refinement without rate loss.

We also provide examples for which the (optimal) refinement that achieves the nonscalable rate-distortion bound in both layers is not everywhere additive. We use the Bernoulli source with continuous reproduction alphabet and square-error distortion measure, as the first example for this phenomenon. The second example involves a three-symbol discrete source with Hamming distortion measure, where addition is modulo 3. (For discrete sources with Hamming distortion measure, it was shown in [6] that successive refinability is possible everywhere without rate loss.)

The organization of the rest of this work is as follows. In the next section, we discuss the preliminaries. The forward and converse additive refinement rate-distortion regions are given in Section III. In Section IV, we analyze the no-excess-rate case, i.e., the necessary and sufficient conditions for the additive refinement bound to coincide with the general scalable coding bound. Finally, in Section V, we analyze several example sources.

## II. PRELIMINARIES

Let  $\{X_t\}_{t=1}^{\infty}$ ,  $X_t \in \mathcal{X}$ , be a discrete memoryless source (DMS) with generic distribution  $P_X(x)$ . Let  $\mathcal{Y}$  denote the reproduction alphabet. Assume a single-letter distortion measure  $d: \mathcal{X} \times \mathcal{Y} \rightarrow [0, \infty)$ , i.e.,

$$d(x^n, y^n) = \frac{1}{n} \sum_{t=1}^n d(x_t, y_t). \quad (1)$$

A block code  $(f, g)$  consists of an encoding function

$$f: \mathcal{X}^n \rightarrow \mathcal{M}$$

which maps the source to index set  $\mathcal{M}$ , and a decoding function

$$g: \mathcal{M} \rightarrow \mathcal{Y}^n.$$

A rate-distortion pair  $(R, D)$  is called *achievable* if for every  $\delta > 0$  and sufficiently large  $n$ , there exists a block code  $(f, g)$  such that

$$\frac{1}{n} \log |\mathcal{M}| \leq R + \delta$$

and

$$E\{d(X^n, g(f(X^n)))\} \leq D + \delta.$$

The fundamental result of rate-distortion theory [2]–[4], [14] is that the rate-distortion function indicating the minimum achievable rate  $R$  for prescribed distortion  $D$  is given by minimizing the mutual information  $I(X; Y)$  over all conditional distributions  $P_{Y|X}(y|x)$  that maintain the prescribed distortion, i.e.,

$$R(D) = \min_{P_{Y|X}(y|x): E\{d(X, Y)\} \leq D} I(X; Y). \quad (2)$$

Let us consider the extension of this scenario to two-layer scalable source coding, where the system consists of encoders

$$\begin{aligned} f_1: \mathcal{X}^n &\rightarrow \mathcal{M}_1 \\ f_2: \mathcal{X}^n &\rightarrow \mathcal{M}_2 \end{aligned}$$

and decoders

$$\begin{aligned} g_1: \mathcal{M}_1 &\rightarrow \mathcal{Y}^n \\ g_2: \mathcal{M}_1 \times \mathcal{M}_2 &\rightarrow \mathcal{Y}^n. \end{aligned}$$

A quadruple  $(R_1, R_2, D_1, D_2)$  is called *scalably achievable* if for every  $\delta > 0$  and sufficiently large  $n$ , there exists a block code  $(f_1, f_2, g_1, g_2)$  such that

$$\begin{aligned} \frac{1}{n} \log |\mathcal{M}_1| &\leq R_1 + \delta \\ \frac{1}{n} \log |\mathcal{M}_1| |\mathcal{M}_2| &\leq R_1 + R_2 + \delta \\ E\{d(X^n, g_1(f_1(X^n)))\} &\leq D_1 + \delta \\ E\{d(X^n, g_2(f_1(X^n), f_2(X^n)))\} &\leq D_2 + \delta. \end{aligned}$$

Rimoldi [12] characterized the minimum scalably achievable total rate  $R_1 + R_2$  for prescribed  $(D_1, D_2, R_1)$  as

$$R(D_1, D_2, R_1) = \min_{\substack{P_{Y_1, Y_2|X}(y_1, y_2|x): \\ I(X; Y_1) \leq R_1 \\ E\{d(X, Y_1)\} \leq D_1 \\ E\{d(X, Y_2)\} \leq D_2}} I(X; Y_1, Y_2). \quad (3)$$

The decoding scheme  $(g_1, g_2)$  in the above scalable scenario is the most general decoding strategy, since the decoders are not constrained in how they process all the available information.

Let us next consider an *additive* decoding scheme, where the decoder functions are constrained to be of the form

$$g_1(i_1) = h_1(i_1) \quad (4)$$

$$g_2(i_1, i_2) = h_1(i_1) + h_2(i_2) \quad (5)$$

where  $h_1(i_1)$  and  $h_2(i_2)$  are independent *stage decoders*. We require that  $(\mathcal{Y}, +)$  form an Abelian (commutative) group,<sup>1</sup> and  $+$  operate “per letter” on vectors as usual. This new constraint imposed on the decoders brings significant storage savings at the reproduction site. The most general decoding scheme needs a total lookup table storage of  $|\mathcal{M}_1|(1+|\mathcal{M}_2|)$  vectors, whereas the additive decoding scheme requires only  $|\mathcal{M}_1| + |\mathcal{M}_2|$ .

*Definition 1:* The quadruple  $(R_1, R_2, D_1, D_2)$  is **additively  $(\alpha, \beta)$ -achievable**, if for large  $n$ , there exists an additive refinement coder  $(f_1, f_2, h_1, h_2)$  with

$$\frac{1}{n} \log |\mathcal{M}_1| \leq R_1 + \alpha \quad (6)$$

$$\frac{1}{n} \log |\mathcal{M}_2| \leq R_2 + \alpha \quad (7)$$

$$E\{d(X^n, h_1(f_1(X^n)))\} \leq D_1 + \beta \quad (8)$$

$$E\{d(X^n, h_1(f_1(X^n)) + h_2(f_2(X^n)))\} \leq D_2 + \beta. \quad (9)$$

*Definition 2:* The quadruple  $(R_1, R_2, D_1, D_2)$  is called **additively achievable**, if it is  $(\alpha, \beta)$ -achievable for all  $\alpha > 0$  and  $\beta > 0$ .

The entire achievable region, denoted by  $\mathcal{Q}$ , is the set of all additively achievable quadruples  $(R_1, R_2, D_1, D_2)$ .

We close this section by recalling the fundamental property of strong typicality, which will be extensively used in the main results section. A vector  $z^n \in \mathcal{Z}^n$  is said to be strongly  $\epsilon$ -typical with respect to random variable  $Z \sim P_Z$  if

$$\left| P_Z(a) - \frac{1}{n} N(a|z^n) \right| < \epsilon, \quad \forall a \in \mathcal{Z}$$

where  $N(a|z^n)$  denotes the number of occurrences of symbol  $a$  in  $z^n$ . The strongly  $\epsilon$ -typical set of  $Z$ , denoted by  $T_\epsilon^n(Z)$ , is the set of all vectors that are strongly  $\epsilon$ -typical with respect to  $Z$ . The strong typicality property is generalized in a straightforward manner to handle jointly distributed multiple random variables. See [4] for a detailed discussion of types and typical sets.

### III. MAIN RESULTS

The first theorem defines an achievable region of quadruples  $(R_1, R_2, D_1, D_2)$ . In other words, it provides sufficient conditions for achievability.

*Theorem 1:* The quadruple  $(R_1, R_2, D_1, D_2)$  is additively  $(\alpha, \beta)$ -achievable if there exist random variables  $\hat{X}_1$  and  $\hat{X}_2$ , jointly distributed with source variable  $X$ , such that

$$I(X; \hat{X}_1) \leq R_1 + \alpha$$

$$I(X; \hat{X}_2) \leq R_2 + \alpha$$

<sup>1</sup>As a matter of fact, a more relaxed condition that is sufficient for the validity of our results is that  $H(A, B) = H(A, A + B)$  for any pair of jointly distributed random variables  $A$  and  $B$ . However, practical examples of addition usually form an Abelian group.

$$I(X; \hat{X}_1, \hat{X}_2) + I(\hat{X}_1; \hat{X}_2) \leq R_1 + R_2 + 2\alpha$$

$$E\{d(X, \hat{X}_1)\} \leq D_1 + \beta$$

$$E\{d(X, \hat{X}_1 + \hat{X}_2)\} \leq D_2 + \beta. \quad (10)$$

*Proof:* The proof follows similar lines as that of the El Gamal-Cover achievability region for the multiple descriptions problem [7]. Let  $P_{\hat{X}_1, \hat{X}_2|X}(\hat{x}_1, \hat{x}_2|x)$  be the conditional distribution satisfying (10). Let the vector elements of the random sets  $\mathcal{C}_1 = \{\hat{x}_1^n(i_1): i_1 \in \mathcal{M}_1\}$  and  $\mathcal{C}_2 = \{\hat{x}_2^n(i_2): i_2 \in \mathcal{M}_2\}$  be drawn independently and according to uniform distributions over  $T_\epsilon^n(\hat{X}_1)$  and  $T_\epsilon^n(\hat{X}_2)$ , respectively.

*Encoding:* Given  $x^n \in \mathcal{X}^n$ , the encoder searches over  $(i_1, i_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ . If it finds an  $(i_1, i_2)$  such that

$$(x^n, \hat{x}_1^n(i_1), \hat{x}_2^n(i_2)) \in T_\epsilon^n(X, \hat{X}_1, \hat{X}_2)$$

then it stops the search and declares  $f_1(x^n) = i_1$ ,  $f_2(x^n) = i_2$ . Otherwise, it sends  $f_1(x^n) = f_2(x^n) = 1$ .

*Decoding:* The decoders simply perform  $h_1(i_1) = \hat{x}_1^n(i_1)$  and  $h_2(i_2) = \hat{x}_2^n(i_2)$ .

*Expected Distortion:* By definition of the strongly  $\epsilon$ -typical sets, if  $(x^n, \hat{x}_1^n, \hat{x}_2^n) \in T_\epsilon^n(X, \hat{X}_1, \hat{X}_2)$ , then

$$\begin{aligned} d(x^n, \hat{x}_1^n) &= \frac{1}{n} \sum_{t=1}^n d(x_t, \hat{x}_{1t}) \\ &= \sum_{(a,b,c) \in \mathcal{X} \times \hat{\mathcal{X}} \times \hat{\mathcal{X}}} \frac{1}{n} N(a, b, c|x^n, \hat{x}_1^n, \hat{x}_2^n) d(a, b) \\ &\leq \sum_{(a,b,c) \in \mathcal{X} \times \hat{\mathcal{X}} \times \hat{\mathcal{X}}} (P_{X, \hat{X}_1, \hat{X}_2}(a, b, c) + \epsilon) d(a, b) \\ &\leq E\{d(X, \hat{X}_1)\} + \epsilon c_1 \\ &\leq D_1 + \beta + \epsilon c_1 \end{aligned}$$

where  $c_1$  is a constant determined by the distortion measure. Similarly

$$d(x^n, \hat{x}_1^n + \hat{x}_2^n) \leq E\{d(X, \hat{X}_1 + \hat{X}_2)\} + \epsilon c_2 \leq D_2 + \beta + \epsilon c_2$$

where  $c_2$  only depends on the distortion measure. Now, denote by  $\mathcal{E}$  the “encoding failure” event where for all  $i_1 \in \mathcal{M}_1$  and  $i_2 \in \mathcal{M}_2$ ,  $(x^n, \hat{x}_1^n(i_1), \hat{x}_2^n(i_2)) \notin T_\epsilon^n(X, \hat{X}_1, \hat{X}_2)$ . From the preceding discussion, it follows that

$$E\{d(X^n, h_1(f_1(X^n)))\} \leq D_1 + \beta + \epsilon c_1 + \Pr[\mathcal{E}] \max_{a,b} d(a, b)$$

and

$$\begin{aligned} E\{d(X^n, h_1(f_1(X^n)) + h_2(f_2(X^n)))\} \\ \leq D_2 + \beta + \epsilon c_2 + \Pr[\mathcal{E}] \max_{a,b} d(a, b). \end{aligned}$$

The probability  $\Pr[\mathcal{E}]$  was shown in [8] to vanish as  $n \rightarrow \infty$  if

$$\frac{1}{n} \log |\mathcal{M}_1| > I(X; \hat{X}_1) + \epsilon c_3$$

$$\frac{1}{n} \log |\mathcal{M}_2| > I(X; \hat{X}_2) + \epsilon c_4$$

$$\frac{1}{n} \log |\mathcal{M}_1| |\mathcal{M}_2| > I(X; \hat{X}_1, \hat{X}_2) + I(\hat{X}_1; \hat{X}_2) + \epsilon c_5.$$

Therefore, setting  $\log |\mathcal{M}_i| = n(R_i + \alpha)$  for  $i = 1, 2$  and letting  $\epsilon \rightarrow 0$  establishes the result.  $\square$

We denote by  $\mathcal{Q}_{\text{in}}$  the region of all quadruples  $(R_1, R_2, D_1, D_2)$  for which there exist random variables  $\hat{X}_1$  and  $\hat{X}_2$  satisfying (10) for all  $\alpha > 0, \beta > 0$ . Obviously,  $\mathcal{Q}_{\text{in}}$  is an inner bound to the achievability region  $\mathcal{Q}$ , i.e.,  $\mathcal{Q}_{\text{in}} \subset \mathcal{Q}$ . Given the similarity of both the method of proof and the resulting conditions to those of [7], and recalling further the fact that the El Gamal–Cover region is not tight [15], we conjecture that  $\mathcal{Q}_{\text{in}} \neq \mathcal{Q}$ .

Turning to converse results, a straightforward observation is that the general (not necessarily additive) scalably achievable region characterized by Rimoldi's  $R(D_1, D_2, R_1)$  function (3) automatically defines a converse region for the additive refinement problem. Let us denote Rimoldi's region by  $\mathcal{Q}_R$ . The following lemma provides an alternative characterization of  $\mathcal{Q}_R$ , which is more convenient for additive scalability considerations.

*Lemma 1:* Given  $D_1, D_2$ , and  $R_1$ , the minimum overall rate  $R_1 + R_2$  required for a quadruple  $(R_1, R_2, D_1, D_2)$  to be scalably achievable is given by

$$R(D_1, D_2, R_1) = \min_{\substack{I(X; \hat{X}_1) \leq R_1 \\ E\{d(X, \hat{X}_1)\} \leq D_1 \\ E\{d(X, \hat{X}_1 + \hat{X}_2)\} \leq D_2}} I(X; \hat{X}_1, \hat{X}_2) \quad (11)$$

where the minimization is over  $P_{\hat{X}_1, \hat{X}_2|X}(\hat{x}_1, \hat{x}_2|x)$ .

*Proof:* Since  $(\hat{X}, +)$  is an Abelian group, observe that

$$I(X; \hat{X}_1, \hat{X}_2) = I(X; \hat{X}_1, \hat{X}_1 + \hat{X}_2).$$

Therefore, the proof follows from Rimoldi's result [12] after making the transformations  $Y_1 = \hat{X}_1$  and  $Y_2 = \hat{X}_1 + \hat{X}_2$ .  $\square$

In the next theorem, we prove a stronger converse result.

*Theorem 2:* If the quadruple  $(R_1, R_2, D_1, D_2)$  is additively  $(\alpha, \beta)$ -achievable, then there exist random variables  $\hat{X}_1$  and  $\hat{X}_2$ , jointly distributed with source variable  $X$ , such that

$$\begin{aligned} I(X; \hat{X}_1) &\leq R_1 + \alpha \\ I(X; \hat{X}_2) &\leq R_2 + \alpha \\ I(X; \hat{X}_1, \hat{X}_2) &\leq R_1 + R_2 + 2\alpha \\ Ed(X, \hat{X}_1) &\leq D_1 + \beta \\ Ed(X, \hat{X}_1 + \hat{X}_2) &\leq D_2 + \beta. \end{aligned} \quad (12)$$

*Proof:* Since  $(R_1, R_2, D_1, D_2)$  is additively  $(\alpha, \beta)$ -achievable, for large enough  $n$ , we have for  $i = 1, 2$

$$\begin{aligned} n(R_i + \alpha) &\geq \log |\mathcal{M}_i| \\ &\geq H(\hat{X}_i^n) \\ &\geq I(X^n; \hat{X}_i^n) \\ &\geq \sum_{t=1}^n I(X_t; \hat{X}_{it}) \end{aligned} \quad (13)$$

where the last inequality follows from the independence of the samples  $X_t$ . Moreover

$$\begin{aligned} n(R_1 + R_2 + 2\alpha) &\geq \log |\mathcal{M}_1| + \log |\mathcal{M}_2| \\ &\geq H(\hat{X}_1^n) + H(\hat{X}_2^n) \end{aligned} \quad (14)$$

$$\begin{aligned} &\geq H(\hat{X}_1^n, \hat{X}_2^n) \\ &\geq I(X^n; \hat{X}_1^n, \hat{X}_2^n) \\ &\geq \sum_{t=1}^n I(X_t; \hat{X}_{1t}, \hat{X}_{2t}). \end{aligned} \quad (15)$$

Also, by definition of additive achievability

$$n(D_1 + \beta) \geq \sum_{t=1}^n E\{d(X_t, \hat{X}_{1t})\} \quad (16)$$

$$n(D_2 + \beta) \geq \sum_{t=1}^n E\{d(X_t, \hat{X}_{1t} + \hat{X}_{2t})\}. \quad (17)$$

Now, let

$$R^*(D_1, D_2, R_1, R_2) = \min_{\substack{I(X; \hat{X}_1) \leq R_1 \\ I(X; \hat{X}_2) \leq R_2 \\ E\{d(X, \hat{X}_1)\} \leq D_1 \\ E\{d(X, \hat{X}_1 + \hat{X}_2)\} \leq D_2}} I(X; \hat{X}_1, \hat{X}_2). \quad (18)$$

Then from (15)

$$\begin{aligned} n(R_1 + R_2 + 2\alpha) &\geq \sum_{t=1}^n R^*(E\{d(X_t, \hat{X}_{1t})\}, \\ &\quad E\{d(X_t, \hat{X}_{1t} + \hat{X}_{2t})\}, I(X_t; \hat{X}_{1t}), I(X_t; \hat{X}_{2t})). \end{aligned}$$

By the (obvious) convexity of  $R^*(\cdot, \cdot, \cdot, \cdot)$ , using (13), (16), and (17), we obtain

$$R_1 + R_2 + 2\alpha \geq R^*(D_1 + \beta, D_2 + \beta, R_1 + \alpha, R_2 + \alpha)$$

which, with the definition of  $R^*$  in (18), implies the theorem.  $\square$

We denote by  $\mathcal{Q}_{\text{out}}$  the outer bound to  $\mathcal{Q}$  characterized by (18), i.e., the region of all  $(R_1, R_2, D_1, D_2)$  such that  $R_1 + R_2 \geq R^*(D_1, D_2, R_1, R_2)$ . From the foregoing discussion, it is clear that

$$\mathcal{Q}_{\text{in}} \subset \mathcal{Q} \subset \mathcal{Q}_{\text{out}} \subset \mathcal{Q}_R \quad (19)$$

where  $\mathcal{Q}_{\text{out}} \subset \mathcal{Q}_R$  follows from (11) and (18).

#### IV. OPTIMALITY OF ADDITIVE REFINEMENT

In this section, we consider the question whether it is possible for an additive refinement coder to achieve the general (not necessarily additive) scalable coding bound, i.e.,  $R_1 + R_2 = R(D_1, D_2, R_1)$ . There is important practical motivation behind this question. If there exists an additive refinement coder achieving  $R_1 + R_2 = R(D_1, D_2, R_1)$ , we can significantly reduce the codebook storage without sacrificing (asymptotic) performance relative to unconstrained scalable coding. Not surprisingly, to answer this question, we adopt an approach very similar to that introduced in the discussion of no-excess-rate in multiple descriptions [1].

If  $(R_1, R_2, D_1, D_2)$  is additively  $(\alpha, \beta)$ -achievable, and  $R_1 + R_2 = R(D_1, D_2, R_1)$ , then from (14)

$$n(R(D_1, D_2, R_1) + 2\alpha) \geq H(\hat{X}_1^n) + H(\hat{X}_2^n).$$

Also, from (13), (15)–(17), and using the fact that  $(\hat{X}, +)$  is an Abelian group

$$\begin{aligned} \frac{1}{n} H(\hat{X}_1^n, \hat{X}_2^n) &= \frac{1}{n} H(\hat{X}_1^n, \hat{X}_1^n + \hat{X}_2^n) \\ &\geq R(D_1 + \beta, D_2 + \beta, R_1 + \alpha). \end{aligned}$$

Hence,

$$\begin{aligned} I(\hat{X}_1^n; \hat{X}_2^n) &= H(\hat{X}_1^n) + H(\hat{X}_2^n) - H(\hat{X}_1^n, \hat{X}_2^n) \\ &\leq n[R(D_1, D_2, R_1) + 2\alpha \\ &\quad - R(D_1 + \beta, D_2 + \beta, R_1 + \alpha)]. \end{aligned}$$

It follows from the continuity of  $R(D_1, D_2, R_1)$  in all of its arguments that, given fixed  $\epsilon$ , we can choose  $\alpha$  and  $\beta$  small enough to ensure that

$$I(\hat{X}_1^n; \hat{X}_2^n) \leq \epsilon n. \quad (20)$$

In other words,  $\hat{X}_1^n$  and  $\hat{X}_2^n$  are almost independent. Notice that we need a single-letter characterization of all additively achievable  $(R_1, R_2, D_1, D_2)$  such that  $R_1 + R_2 = R(D_1, D_2, R_1)$ , and hence, as it stands, (20) is not strong enough. However, if we proved  $I(\hat{X}_{1t}; \hat{X}_{2t}) \leq \epsilon$ , then we would be able to derive a tighter converse than  $\mathcal{Q}_{\text{out}}$ , by adding

$$I(\hat{X}_1; \hat{X}_2) = 0$$

to the necessary conditions in (12), when  $\alpha \rightarrow 0, \beta \rightarrow 0$ . In other words, we would require, as the necessary condition for additive achievability of  $(R_1, R_2, D_1, D_2)$ , the existence of random variables  $\hat{X}_1$  and  $\hat{X}_2$  such that

$$\begin{aligned} I(X; \hat{X}_1) &\leq R_1 \\ I(X; \hat{X}_2) &\leq R_2 \\ I(\hat{X}_1; \hat{X}_2) &= 0 \\ I(X; \hat{X}_1, \hat{X}_2) &\leq R_1 + R_2 = R(D_1, D_2, R_1) \\ Ed(X, \hat{X}_1) &\leq D_1 \\ Ed(X, \hat{X}_1 + \hat{X}_2) &\leq D_2. \end{aligned} \quad (21)$$

Note further that, random variables  $\hat{X}_1$  and  $\hat{X}_2$  satisfying (21) also satisfy (10). Therefore, (21) also constitutes a sufficient condition for additive achievability of  $(R_1, R_2, D_1, D_2)$ . In the proof of the next theorem which formalizes this discussion, it is instead shown using (20) that  $\hat{X}_{1t}$  and  $\hat{X}_{2t}$  are conditionally independent given relatively few sample components from  $\hat{X}_1^n$  and  $\hat{X}_2^n$ . This result is then used to generate new random variables satisfying (21).

*Theorem 3:* The quadruple  $(R_1, R_2, D_1, D_2)$ , where  $R_1 + R_2 = R(D_1, D_2, R_1)$ , is additively achievable if and only if

there exist random variables  $\hat{X}_1$  and  $\hat{X}_2$ , jointly distributed with source variable  $X$ , such that (21) holds.

The proof is almost identical to Ahlswede's proof for the no-excess-rate case for the multiple descriptions problem [1], and is therefore relegated to Appendix A for completeness without disrupting the flow of the text.

## V. EXAMPLES FOR ADDITIVE REFINEMENT WITH OR WITHOUT RATE LOSS

In this section, we analyze different memoryless sources and investigate values of  $D_1$  and  $D_2$  where additive refinement imposes excess rate expenditure relative to unconstrained refinement. Since it is not easy to find closed-form expressions for the conditional distributions achieving  $R(D_1, D_2, R_1)$ , we focus on sources that are successively refinable without rate loss, i.e., the case where  $R(D_1, D_2, R(D_1)) = R(D_2)$ . For this special case, it is clear from (11) that one can readily find random variables  $\hat{X}_1$  and  $\hat{X}_2$  such that

$$\begin{aligned} I(X; \hat{X}_1) &= R(D_1) \\ I(X; \hat{X}_1, \hat{X}_2) &= R(D_2) \\ Ed(X, \hat{X}_1) &\leq D_1 \\ Ed(X, \hat{X}_1 + \hat{X}_2) &\leq D_2. \end{aligned}$$

If, in addition,  $\hat{X}_1$  and  $\hat{X}_2$  are independent, then

$$\begin{aligned} R(D_2) &= I(X; \hat{X}_1, \hat{X}_2) + I(\hat{X}_1; \hat{X}_2) \\ &= I(X; \hat{X}_1) + I(X; \hat{X}_2) + I(\hat{X}_1; \hat{X}_2|X) \\ &\geq R(D_1) + I(X; \hat{X}_2). \end{aligned}$$

Hence  $I(X; \hat{X}_2) \leq R(D_2) - R(D_1) = R_2$ , and (21) is automatically satisfied.

Although we derived our results for DMSs, the results can be extended to memoryless sources with abstract alphabets, under the usual assumption that there exists a "reference letter"  $a$  such that  $E\{d(X, a)\} = d^* < \infty$ . Therefore, in the following subsections, we also discuss continuous alphabet sources satisfying this extra constraint.

We begin by analyzing continuous sources with tight SLB [2], under square-error and absolute-error distortion measures. For both measures, it is known that tightness of the SLB for  $D = D_0$  implies tightness of it for any  $D < D_0$ . Hence, there exists a *critical distortion*  $D_c$  such that the SLB is tight if and only if  $D \leq D_c$  [13]. We prove that for continuous sources with  $D_2 \leq D_1 \leq D_c$ , successive refinement without rate loss is achievable not only generally [6], but also in the additive-decoding sense. We then discuss the Bernoulli source under the square-error distortion measure, where the reproduction alphabet is the real line. We show that the optimal refinement is nowhere additive, although the source is successively refinable without rate loss for all distortion values. Finally, we investigate a discrete source under the Hamming distortion, for which

we show that additive refinement is optimal if and only if  $D_1$  is below a specified threshold.

#### A. Continuous Sources at Subcritical Distortion Under Square-Error and Absolute-Error Distortion Measures

Let  $\mathcal{X} = \hat{\mathcal{X}} = \mathcal{R}$ , and let  $X$  have a probability density function (pdf)  $P_X(x)$ . Assume that the critical distortion  $D_c$  is positive, and let  $D_2 \leq D_1 \leq D_c$ . From the discussion in [6], it follows that

$$R(D_1, D_2, R(D_1)) = R(D_2)$$

i.e., successive refinement without rate loss is possible, for both square-error and absolute-error distortion measures. We show in what follows a stronger result for both cases: for all  $D_2 \leq D_1 \leq D_c$ , additive refinement without rate loss is achievable.

For the square-error case, the random variables  $Y_1$  and  $Y_2$ , achieving  $R(D_1)$  and  $R(D_2)$ , respectively, also satisfy

$$X = Y_2 + N_2$$

$$Y_2 = Y_1 + N_1$$

where  $N_1 \sim \mathcal{N}(0, D_1 - D_2)$  and  $N_2 \sim \mathcal{N}(0, D_2)$ . This is the well-known “backward-channel” interpretation [2]. The noise terms  $N_1$  and  $N_2$  are independent from each other and from  $Y_1$  and  $Y_2$ , respectively. Introducing the transformations  $Y_1 = \hat{X}_1$  and  $Y_2 = \hat{X}_1 + \hat{X}_2$ , we obtain

$$X = \hat{X}_1 + \hat{X}_2 + N_2$$

$$\hat{X}_2 = N_1.$$

It easily follows that  $I(\hat{X}_1; \hat{X}_2) = 0$ , and the rest of the conditions in (21) are automatically satisfied.

Similarly, for the absolute-error case, we have

$$X = Y_2 + Z_2$$

$$Y_2 = Y_1 + Z_1$$

where

$$P_{Z_2}(z_2) = \frac{1}{2D_2} \exp\left\{-\frac{|z_2|}{D_2}\right\}$$

i.e.,  $Z_2$  has a Laplacian distribution, and

$$P_{Z_1}(z_1) = \frac{D_2^2}{D_1^2} \delta(z_1) + \left(1 - \frac{D_2^2}{D_1^2}\right) \frac{1}{2D_1} \exp\left\{-\frac{|z_1|}{D_1}\right\}.$$

As in the square-error case, the noise terms  $Z_1$  and  $Z_2$  are independent from each other and from  $Y_1$  and  $Y_2$ , respectively. Upon introducing the transformations  $Y_1 = \hat{X}_1$  and  $Y_2 = \hat{X}_1 + \hat{X}_2$ , it follows that  $I(\hat{X}_1; \hat{X}_2) = 0$ , and therefore (21) is satisfied.

#### B. Bernoulli Source Under Square-Error Distortion Measure

Let  $X$  be a Bernoulli source, i.e., distributed with  $P_X(x) = (1-p)\delta(x) + p\delta(x-1)$ , where  $\delta(x)$  denotes the Dirac delta function. Without loss of generality, we can assume that  $p \geq 0.5$ . Let  $\hat{\mathcal{X}} = \mathcal{R}$ , i.e., the reproduction is allowed to be positive everywhere on the real line, and let  $d(x, y) = (x-y)^2$ .

Now, we present a lemma which introduces a parameter characterizing the optimal reproduction and backward channel distributions, and the attained distortion. The proof will be presented in Appendix B.

*Lemma 2:* For  $0 \leq D < p(1-p)$ , the optimal reproduction distribution, achieving the rate distortion function  $R(D)$ , and the corresponding backward channel, are, respectively, given by

$$P_Y(y) = q\delta(y-a) + (1-q)\delta(y-1+a) \quad (22)$$

and

$$P_{X|Y}(x|y) = \begin{cases} 1-a, & x=0, y=a \text{ or } x=1, y=1-a \\ a, & x=1, y=a \text{ or } x=0, y=1-a \end{cases} \quad (23)$$

where

$$q = \frac{1-p-a}{1-2a}$$

and  $a$  is determined by  $D = a(1-a)$  and  $0 \leq a < 1-p$ .

Note that the distortion values of interest are within the interval  $0 \leq D < p(1-p)$ , because  $R(D) = 0$  for all  $D > p(1-p)$ . From [6], successive refinability without rate loss from  $D_1$  to  $D_2$  with  $D_1 > D_2$  is possible if and only if  $X - Y_2 - Y_1$  form a Markov chain, i.e., if and only if one can find a legitimate  $P_{Y_2|Y_1}(y_2|y_1)$  satisfying

$$P_{X|Y_1}(x|y_1) = \int P_{Y_2|Y_1}(y_2|y_1) P_{X|Y_2}(x|y_2) dy_2$$

where  $P_{X|Y_1}(x|y_1)$  and  $P_{X|Y_2}(x|y_2)$  correspond to  $D_1$  and  $D_2$ , respectively. But this condition is equivalent to the existence of a matrix  $\mathbf{P}$  with nonnegative entries, where

$$\begin{bmatrix} 1-b & b \\ b & 1-b \end{bmatrix} \cdot \mathbf{P} = \begin{bmatrix} 1-a & a \\ a & 1-a \end{bmatrix}$$

with  $D_1 = a(1-a)$ ,  $D_2 = b(1-b)$ , and  $0 \leq a, b < 1-p$ . Solving for  $\mathbf{P}$ , we get

$$\mathbf{P} = \frac{1}{1-2b} \begin{bmatrix} 1-a-b & a-b \\ a-b & 1-a-b \end{bmatrix}$$

whose entries are obviously positive, since  $D_1 > D_2$  implies  $a > b$ . In other words,  $P_{Y_1, Y_2}(y_1, y_2)$  is given by

$$\begin{aligned} & P_{Y_1, Y_2}(y_1, y_2) \\ &= \frac{1}{(1-2b)(1-2a)} \\ & \cdot [(1-a-b)(1-p-a)\delta(y_1-a, y_2-b) \\ & + (a-b)(p-a)\delta(y_1-1+a, y_2-b) \\ & + (a-b)(1-p-a)\delta(y_1-a, y_2-1+b) \\ & + (1-a-b)(p-a)\delta(y_1-1+a, y_2-1+b)]. \end{aligned}$$

The only distribution  $P_{\hat{X}_1, \hat{X}_2|X}(\hat{x}_1, \hat{x}_2|x)$  that could satisfy (21) is found by applying the transformations  $\hat{X}_1 = Y_1$  and  $\hat{X}_2 = Y_2 - Y_1$ . It immediately follows that this choice of  $\hat{X}_1, \hat{X}_2$  does not satisfy  $I(\hat{X}_1; \hat{X}_2) = 0$ .

#### C. A Discrete Alphabet Source With Hamming Distortion Measure

Let  $\mathcal{X} = \hat{\mathcal{X}} = \{0, 1, 2\}$  and let “+” be defined as modulo 3 summation. Also, let  $P_X(x) = [p_0, p_1, p_2]$  such that  $p_0 \geq p_1 \geq p_2$ . We know that this source is successively refinable without rate loss in the general sense [6], i.e.,  $R(D_1, D_2, R(D_1)) = R(D_2)$ . Using the transformations  $y_1 = \hat{x}_1$  and  $y_2 = \hat{x}_1 + \hat{x}_2$

on the conditional distribution  $P_{Y_1, Y_2|X}(y_1, y_2|x)$  that achieves the rate-distortion bound in both layers, provided in [6], we observe that the only distribution  $P_{\hat{X}_1, \hat{X}_2|X}(\hat{x}_1, \hat{x}_2|x)$  that could satisfy (21) is given by

$$P_{\hat{X}_1, \hat{X}_2|X}(\hat{x}_1, \hat{x}_2|x) = P_{\hat{X}_1}(\hat{x}_1)P_{\hat{X}_2|\hat{X}_1}(\hat{x}_2|\hat{x}_1)P_{X|\hat{X}_1, \hat{X}_2}(x|\hat{x}_1, \hat{x}_2)$$

where

$$P_{\hat{X}_1}(\hat{x}_1) = \frac{|P_X(\hat{x}_1) - \alpha_1|^+}{\sum_z |P_X(z) - \alpha_1|^+}$$

is the first layer reproduction distribution

$$P_{\hat{X}_2|\hat{X}_1}(\hat{x}_2|\hat{x}_1) = \begin{cases} \frac{1 - D_1 - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 = 0 \\ \frac{\max[\alpha_2, \min(\alpha_1, P_X(\hat{x}_1 + \hat{x}_2))] - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 \neq 0 \end{cases}$$

is the second-layer distribution conditioned on the first-layer reproduction, and finally

$$P_{X|\hat{X}_1, \hat{X}_2}(x|\hat{x}_1, \hat{x}_2) = \begin{cases} 1 - D_2, & x = \hat{x}_1 + \hat{x}_2 \\ \min(\alpha_2, P_X(x)), & x \neq \hat{x}_1 + \hat{x}_2 \end{cases}$$

is the backward channel. Here,  $\alpha_1$  and  $\alpha_2$  are uniquely determined by  $D_1$  and  $D_2$ , respectively, according to

$$D_i = -\alpha_i + \sum_x \min(\alpha_i, P_X(x)), \quad i = 1, 2. \quad (24)$$

Note that when  $D_1 = 1 - p_0$ , i.e., when  $\alpha_1 \geq p_1$ , the independence condition  $I(\hat{X}_1; \hat{X}_2) = 0$  is trivially satisfied, since  $P_{\hat{X}_1}(\hat{x}_1) > 0$  only for  $\hat{x}_1 = 0$ . This is not surprising, since any two-layer tree-structured decoding scheme with 0 rate at the first layer is implementable by additive decoding. We analyze the remaining nontrivial cases,  $p_1 > \alpha_1 > \alpha_2 > p_2$ ,  $p_1 > \alpha_1 > p_2 > \alpha_2$ , and  $p_2 \geq \alpha_1 > \alpha_2$ , separately.

**Case I:**  $p_1 > \alpha_1 > \alpha_2 \geq p_2$

For this case,  $P_{\hat{X}_1}(\hat{x}_1) > 0$  only for  $\hat{x}_1 = 0$  or  $\hat{x}_1 = 1$ . Also,

$$P_{\hat{X}_2|\hat{X}_1}(\hat{x}_2|0) = \begin{cases} \frac{1 - D_1 - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 = 0 \\ \frac{\alpha_1 - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 = 1 \\ 0, & \hat{x}_2 = 2 \end{cases}$$

and

$$P_{\hat{X}_2|\hat{X}_1}(\hat{x}_2|1) = \begin{cases} \frac{1 - D_1 - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 = 0 \\ 0, & \hat{x}_2 = 1 \\ \frac{\alpha_1 - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 = 2. \end{cases}$$

Hence  $\hat{X}_1$  and  $\hat{X}_2$  are not independent.

**Case II:**  $p_1 > \alpha_1 > p_2 > \alpha_2$

It is also true for this case that  $P_{\hat{X}_1}(\hat{x}_1) > 0$  only for  $\hat{x}_1 = 0$  or  $\hat{x}_1 = 1$ . Computing the conditionals

$$P_{\hat{X}_2|\hat{X}_1}(\hat{x}_2|0) = \begin{cases} \frac{1 - D_1 - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 = 0 \\ \frac{\alpha_1 - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 = 1 \\ \frac{p_2 - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 = 2 \end{cases}$$

and

$$P_{\hat{X}_2|\hat{X}_1}(\hat{x}_2|1) = \begin{cases} \frac{1 - D_1 - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 = 0 \\ \frac{p_2 - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 = 1 \\ \frac{\alpha_1 - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 = 2 \end{cases}$$

we conclude that  $\hat{X}_1$  and  $\hat{X}_2$  are not independent.

**Case III:**  $p_2 \geq \alpha_1 > \alpha_2$

$P_{\hat{X}_1}(\hat{x}_1) > 0$  for all  $\hat{x}_1$ , and

$$P_{\hat{X}_2|\hat{X}_1}(\hat{x}_2|\hat{x}_1) = \begin{cases} \frac{1 - D_1 - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 = 0 \\ \frac{\alpha_1 - \alpha_2}{1 - D_2 - \alpha_2}, & \hat{x}_2 \neq 0 \end{cases}$$

independent from the value of  $\hat{x}_1$ . Therefore,  $I(\hat{X}_1, \hat{X}_2) = 0$ .

The condition  $\alpha_1 \leq p_2$  can be translated using (24) as  $D_1 \leq 2p_2$ . From the preceding analysis, we conclude that (21) is satisfied for this source if and only if  $D_1 \leq 2p_2$ . We thus have an example of a source that is successively refinable everywhere without any rate loss, but optimal refinement is not additive everywhere.

## VI. CONCLUSION

We analyzed the rate-distortion performance of additive refinement coders. After providing inner and outer bounds for the region of all achievable  $(R_1, R_2, D_1, D_2)$ , we considered the question whether additive refinement coders can be optimal, i.e., whether they can achieve the general scalable coding bound  $R_2 = R(D_1, D_2, R_1)$ . An affirmative answer to this question has an important practical implication: it means that we can significantly reduce the codebook storage without sacrificing performance of the coder. We derived necessary and sufficient conditions for the optimality of additive refinement, and analyzed those conditions for several examples.

### APPENDIX A PROOF OF THEOREM 3

The ‘‘if’’ part of the proof is obvious since (21) implies (10) for all  $\alpha > 0, \beta > 0$ . For the ‘‘only if’’ part, we need two lemmas from [1] known as the *wringing lemmas*.

*Lemma 3:* If  $I(\hat{X}_1^n; \hat{X}_2^n) \leq \sigma$ , then for any  $\delta > 0$ , there exist  $1 \leq t_1, \dots, t_k \leq n$  with  $k \leq \frac{\sigma}{\delta}$ , such that

$$I(\hat{X}_{1t_1}; \hat{X}_{2t_1}|\hat{X}_{1t_1}, \hat{X}_{2t_1}, \dots, \hat{X}_{1t_k}, \hat{X}_{2t_k}) \leq \delta \quad (25)$$

for  $t = 1, \dots, n$ .

If  $(R_1, R_2, D_1, D_2)$  is additively achievable, then it is additively  $(\alpha, \beta)$ -achievable for any  $\alpha > 0$  and  $\beta > 0$ . For sufficiently small  $\alpha$  and  $\beta$  and large  $n$ , we showed that  $I(\hat{X}_1^n; \hat{X}_2^n) \leq \epsilon n$ . Choosing  $\sigma = \epsilon n$ ,  $\delta = \epsilon m$  with  $1 \leq m \leq n$ , we obtain (25) with  $k \leq \frac{n}{m}$ . Now use the abbreviation  $Z = \hat{X}_{1t_1}, \hat{X}_{2t_1}, \dots, \hat{X}_{1t_k}, \hat{X}_{2t_k}$ . For  $i = 1, 2$ , we observe

$$n(R_i + \alpha) \geq H(\hat{X}_i^n) \geq I(X^n; \hat{X}_i^n, Z) - H(Z)$$

and, therefore,

$$R_i + \alpha \geq \frac{1}{n} \sum_{t=1}^n I(X_t; \hat{X}_{it}|Z) - \frac{1}{n} H(Z). \quad (26)$$

Similarly, we can derive

$$R_1 + R_2 + 2\alpha \geq \frac{1}{n} \sum_{t=1}^n I(X_t; \hat{X}_{1t}, \hat{X}_{2t}|Z) - \frac{1}{n} H(Z). \quad (27)$$

Define now for every  $z$  random variables  $\hat{X}_{it}(z)$  and  $X_t(z)$  with distributions

$$\begin{aligned} \Pr[\hat{X}_{it}(z) = \hat{x}_i] &= \Pr[\hat{X}_{it} = \hat{x}_i | Z = z] \\ \Pr[X_t(z) = x] &= \Pr[X_t = x | Z = z] \end{aligned}$$

so that we can write

$$D_1 + \beta \geq \frac{1}{n} \sum_{t=1}^n \sum_z \Pr[Z = z] \cdot E \left\{ d(X_t(z), \hat{X}_{1t}(z)) \right\} \quad (28)$$

$$D_2 + \beta \geq \frac{1}{n} \sum_{t=1}^n \sum_z \Pr[Z = z] \cdot E \left\{ d(X_t(z), \hat{X}_{1t}(z) + \hat{X}_{2t}(z)) \right\}. \quad (29)$$

We also have  $H(Z) \leq k \log |\mathcal{X}|^2$  and hence

$$\frac{1}{n} H(Z) \leq \frac{2}{m} \log |\hat{\mathcal{X}}|. \quad (30)$$

At this stage, we need the second wringing lemma.

*Lemma 4:* Let  $\{X_t\}$  be a DMS and let  $Z$  be a random variable correlated with  $\{X_t\}$ . If we define for  $1 > \rho > 0$

$$\mathcal{Z}(\rho, t) \triangleq \left\{ z: \sum_x |P_{X_t|Z}(x|z) - P_{X_t}(x)| \leq \rho \right\}$$

then for any  $\gamma > 0$  and every  $n$ , there exists a set of indices  $\mathcal{T}(n, \gamma) \subset \{1, \dots, n\}$  such that

$$|\mathcal{T}(n, \gamma)| \geq \frac{\gamma}{1+\gamma} n$$

$$H(X_t|Z) \geq H(X_t) - \frac{1+\gamma}{n} H(Z), \quad \text{for all } t \in \mathcal{T}(n, \gamma)$$

$$P_Z(\mathcal{Z}(\rho, t)) \geq 1 - \frac{1}{\rho} \sqrt{\frac{1+\gamma}{cn} H(Z)}, \quad \text{for all } t \in \mathcal{T}(n, \gamma)$$

where  $c$  is a constant.

Now, define

$$\mathcal{Z}'(\epsilon, t) \triangleq \left\{ z: I(\hat{X}_{1t}(z); \hat{X}_{2t}(z)) \leq \epsilon^{1/5} \right\}$$

and  $\mathcal{Z}(\epsilon, \rho, t) \triangleq \mathcal{Z}'(\epsilon, t) \cap \mathcal{Z}(\rho, t)$ . Also, let  $m = \epsilon^{-2/5}$ . Then, by Chebyshev's inequality and (25)

$$P_Z(\mathcal{Z}'(\epsilon, t)) \geq 1 - \epsilon^{2/5}$$

for  $t = 1, \dots, n$ . Applying the second wringing lemma with

$$1 + \gamma = \frac{c}{2\epsilon^{1/5} \log |\hat{\mathcal{X}}|}$$

and using (30), we obtain

$$P_Z(\mathcal{Z}(\rho, t)) \geq 1 - \rho^{-1} \epsilon^{1/10}$$

for  $t \in \mathcal{T}(n, \gamma)$ . Therefore, it follows that

$$P_Z(\mathcal{Z}(\epsilon, \rho, t)) \geq 1 - \epsilon^{2/5} - \rho^{-1} \epsilon^{1/10} \geq 1 - 2\rho^{-1} \epsilon^{1/10}$$

for  $t \in \mathcal{T}(n, \gamma)$ . Also,  $|\mathcal{T}(n, \gamma)| \geq n\nu(\epsilon)$ , where

$$\nu(\epsilon) \triangleq 1 - 2c^{-1} \epsilon^{1/5} \log |\hat{\mathcal{X}}|.$$

It is easy to see from the preceding equations that as  $\epsilon \rightarrow 0$ ,  $\nu(\epsilon) \rightarrow 1$ ,  $|\mathcal{T}(n, \gamma)| \rightarrow n$ , and  $P_Z(\mathcal{Z}(\epsilon, \rho, t)) \rightarrow 1$ . Motivated by this observation, and using (25)–(30), we obtain the following inequalities:

$$\begin{aligned} & \frac{R_i + \alpha + 2 \log |\hat{\mathcal{X}}| \epsilon^{2/5}}{\nu(\epsilon)(1 - 2\rho^{-1} \epsilon^{1/10})} \\ & \geq \frac{1}{|\mathcal{T}(n, \gamma)|} \sum_{t \in \mathcal{T}(n, \gamma)} \sum_{z \in \mathcal{Z}(\epsilon, \rho, t)} \\ & \quad \cdot \frac{P_Z(z)}{P_Z(\mathcal{Z}(\epsilon, \rho, t))} I(X_t(z); \hat{X}_{it}(z)) \\ & \frac{R_1 + R_2 + 2\alpha + 2 \log |\hat{\mathcal{X}}| \epsilon^{2/5}}{\nu(\epsilon)(1 - 2\rho^{-1} \epsilon^{1/10})} \\ & \geq \frac{1}{|\mathcal{T}(n, \gamma)|} \sum_{t \in \mathcal{T}(n, \gamma)} \sum_{z \in \mathcal{Z}(\epsilon, \rho, t)} \\ & \quad \cdot \frac{P_Z(z)}{P_Z(\mathcal{Z}(\epsilon, \rho, t))} I(X_t(z); \hat{X}_{1t}(z), \hat{X}_{2t}(z)) \\ & \frac{D_1 + \beta}{\nu(\epsilon)(1 - 2\rho^{-1} \epsilon^{1/10})} \\ & \geq \frac{1}{|\mathcal{T}(n, \gamma)|} \sum_{t \in \mathcal{T}(n, \gamma)} \sum_{z \in \mathcal{Z}(\epsilon, \rho, t)} \\ & \quad \cdot \frac{P_Z(z)}{P_Z(\mathcal{Z}(\epsilon, \rho, t))} d(X_t(z), \hat{X}_{1t}(z)) \\ & \frac{D_2 + \beta}{\nu(\epsilon)(1 - 2\rho^{-1} \epsilon^{1/10})} \\ & \geq \frac{1}{|\mathcal{T}(n, \gamma)|} \sum_{t \in \mathcal{T}(n, \gamma)} \sum_{z \in \mathcal{Z}(\epsilon, \rho, t)} \\ & \quad \cdot \frac{P_Z(z)}{P_Z(\mathcal{Z}(\epsilon, \rho, t))} d(X_t(z), \hat{X}_{1t}(z) + \hat{X}_{2t}(z)). \end{aligned}$$

Let  $\epsilon \rightarrow 0$  and observe that the inequalities converge to (13), (15)–(17), respectively, except for an extra expectation with respect to  $Z$ . By a convexity argument, as in the proof of Theorem 2, and also from (25), we establish the result. For a more precise and detailed argument, the reader is referred to [1].



APPENDIX B  
PROOF OF LEMMA 2

Let

$$\beta(a) = \frac{1}{1-2a} \log \frac{1-a}{a} \quad (31)$$

where  $0 < a < 1-p \leq 0.5$ . It is easily verified using differentiation that  $\beta(a)$  is a monotonically decreasing function in the interval  $0 < a < 1-p$ . The range of  $\beta(a)$  in the same interval is  $[\frac{1}{1-2p} \log \frac{1-p}{p}, \infty)$ . We will prove that the tentative solution  $P_Y(y)$  given in (22) satisfies the Kuhn–Tucker optimality conditions [2] with the slope parameter  $\frac{dR(D)}{dD} = -\beta(a)$ . To this end, we write

$$\begin{aligned} f(y) &\triangleq \int \frac{P_X(x)e^{-\beta(a)(x-y)^2}}{\int P_Y(y')e^{-\beta(a)(x-y')^2} dy'} dx \\ &= (1-p)\lambda_0 e^{-\beta(a)y^2} + p\lambda_1 e^{-\beta(a)(1-y)^2} \end{aligned}$$

where

$$\begin{aligned} \lambda_0^{-1} &= q_1 e^{-\beta(a)a^2} + q_2 e^{-\beta(a)(1-a)^2} \\ \lambda_1^{-1} &= q_1 e^{-\beta(a)(1-a)^2} + q_2 e^{-\beta(a)a^2}. \end{aligned}$$

To prove that  $P_Y(y)$  is indeed optimal for the value of  $D$  satisfying  $\frac{dR(D)}{dD} = -\beta(a)$ , it suffices to show  $f(y) \leq 1$ . By substitution, we see

$$(1-p)\lambda_0 = p\lambda_1 = (1-a)e^{\beta(a)a^2} \quad (32)$$

and, hence,

$$f(y) = (1-a)e^{-\beta(a)a^2} \left( e^{-\beta(a)y^2} + e^{-\beta(a)(1-y)^2} \right).$$

By inspection, we observe that  $f(y)$  is symmetric with respect to  $y = 0.5$ , and

$$\begin{aligned} f(a) &= 1 \\ f'(a) &= 0 \\ f''(a) &< 0 \end{aligned}$$

therefore,  $f(y)$  assumes a local maximum of value 1 at  $y = a$  and  $y = 1-a$ . Observing that  $f(y)$  is proportional to the mixture of two Gaussians with the same variance, it follows that  $f(y)$  can assume only two local maxima; therefore, the result  $f(y) \leq 1$  follows.

The backward channel distribution corresponding to  $P_Y(y)$  is given by

$$P_{X|Y}(x|y) = \frac{P_X(x)e^{-\beta(a)(x-y)^2}}{\int P_Y(y')e^{-\beta(a)(x-y')^2}$$

which is to be evaluated only at  $y = a$  and  $y = 1-a$ . Substituting  $P_X(x)$  and  $P_Y(y)$ , we conclude that

$$P_{X|Y}(x|y) = \begin{cases} 1-a, & x=0, y=a \text{ or } x=1, y=1-a \\ a, & x=1, y=a \text{ or } x=0, y=1-a. \end{cases} \quad (33)$$

Given a fixed value of  $a$  with  $0 \leq a \leq 1-p$ , the resulting distortion is

$$D = \sum_{x=0,1} \sum_{y=a,1-a} P_Y(y)P_{X|Y}(x|y)(x-y)^2 = a(1-a). \quad (34)$$

ACKNOWLEDGMENT

The authors wish to thank the anonymous referees for pointing out an error in the statement of Lemma 3.

REFERENCES

- [1] R. Ahlswede, "The rate-distortion region for multiple descriptions without excess rate," *IEEE Trans. Inform. Theory*, vol. IT-31, pp. 721–726, Nov. 1985.
- [2] T. Berger, *Rate Distortion Theory: A Mathematical Basis for Data Compression*. Englewood Cliffs, NJ: Prentice-Hall, 1971.
- [3] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.
- [4] I. Csiszár and J. Körner, *Information Theory: Coding Theorems for Discrete Memoryless Systems*. New York: Academic, 1982.
- [5] M. Effros, "Distortion-rate bounds for fixed- and variable-rate multiresolution source coding," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1887–1910, Sept. 1999.
- [6] W. H. R. Equitz and T. M. Cover, "Successive refinement of information," *IEEE Trans. Inform. Theory*, vol. 37, pp. 269–275, Mar. 1991.
- [7] A. El Gamal and T. M. Cover, "Achievable rates for multiple descriptions," *IEEE Trans. Inform. Theory*, vol. IT-28, pp. 851–857, Nov. 1982.
- [8] A. El Gamal and E. van der Meulen, "A proof of Marton's coding theorem for the discrete memoryless broadcast channel," *IEEE Trans. Inform. Theory*, vol. IT-27, pp. 120–122, Jan. 1981.
- [9] A. Gersho and R. M. Gray, *Vector Quantization and Signal Compression*. Boston, MA: Kluwer, 1992.
- [10] V. Koshchelev, "Hierarchical coding of discrete sources," *Probl. Pered. Inform.*, vol. 16, no. 3, pp. 31–49, 1980.
- [11] —, "An evaluation of the average distortion for discrete schemes of sequential approximation," *Probl. Pered. Inform.*, vol. 17, no. 3, pp. 20–33, 1981.
- [12] B. Rimoldi, "Successive refinement of information: Characterization of the achievable rates," *IEEE Trans. Inform. Theory*, vol. 40, no. 1994, pp. 253–259, Jan..
- [13] K. Rose, "A mapping approach to rate-distortion computation and analysis," *IEEE Trans. Inform. Theory*, vol. 40, pp. 1939–1952, Nov. 1994.
- [14] C. E. Shannon, "Coding theorems for a discrete source with a fidelity criterion," in *IRE Conv. Rec.*, vol. 7, 1959, pp. 142–163.
- [15] Z. Zhang and T. Berger, "New results in binary multiple descriptions," *IEEE Trans. Inform. Theory*, vol. IT-33, pp. 502–521, July 1987.