

Combinatorial Message Sharing for a Refined Multiple Descriptions Achievable Region

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Abstract—This paper presents a new achievable rate-distortion region for the L -channel multiple descriptions problem. Currently, the most popular region for this problem is due to Venkataramani, Kramer and Goyal [3]. Their encoding scheme is an extension of the Zhang-Berger scheme to the L -channel case and includes a combinatorial number of refinement codebooks, one for each subset of the descriptions. All the descriptions also share a single common codeword, which introduces redundancy, but assists in better coordination of the descriptions. This paper proposes a novel encoding technique involving ‘Combinatorial Message Sharing’, where every subset of the descriptions may share a distinct common message. This introduces a combinatorial number of shared codebooks along with the refinement codebooks of [3]. These shared codebooks provide a more flexible framework to trade off redundancy across the messages for resilience to descriptions loss. We derive an achievable rate-distortion region for the proposed technique, and show that it subsumes the achievable region of [3].

Index Terms—Multiple descriptions coding, Source coding, Rate distortion theory

I. INTRODUCTION

The Multiple Descriptions (MD) problem was proposed in the late seventies and has been a challenging problem since then. It has been studied extensively with results ranging from the derivation of asymptotic bounds [1], [2], [3], [4], [5], [6], [7] to practical approaches for multiple descriptions quantizer design [8]. It was originally viewed as a method to cope with channel failures, where multiple source descriptions are generated and sent over different paths. The encoder generates L descriptions for transmission over L available channels. It is assumed that the decoder receives a subset of the descriptions perfectly and the remaining are lost, as shown in Figure 1. The objective of the MD problem is to design the encoders (for each description) and decoders (for each possible received subset of the descriptions), with respect to an overall rate-distortion trade-off. The subtlety of the problem is due to the balance between the full reconstruction quality versus quality of individual descriptions; or noise free quality versus the amount of redundancy across descriptions needed to achieve resilience to descriptions loss.

The most well known achievable region currently known for the L -channel MD problem is due to Venkataramani, Kramer and Goyal (VKG) [3] for general sources and distortion measures. Their encoding scheme builds on the prior work

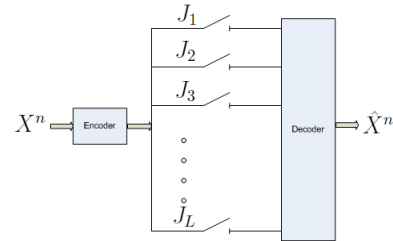


Figure 1. L -Channel Multiple Descriptions Setup : Each description is received error free or is completely lost at the decoder

for the 2-channel case by El-Gamal and Cover (EC) [1] and Zhang and Berger (ZB) [2] and introduces a combinatorial number of refinement codebooks, one for each subset of descriptions. A *single* common codeword is also shared among all the descriptions, which assists in controlling the redundancy across the messages, improving the rate-distortion trade-off. In this paper we present a new encoding scheme involving ‘Combinatorial Message Sharing’ (CMS), where a unique common codeword is sent (shared) in each subset of the descriptions, thereby introducing a combinatorial number of *shared codebooks*, along with the refinement codebooks of [3]. The common codewords enable better coordination between descriptions, providing a refined over-all rate-distortion region. We derive an achievable rate-distortion region for the CMS scheme and show that it subsumes the achievable region due to VKG in [3]. We note that the underlying principle behind CMS has been shown in precursor work to be useful in related applicational contexts of routing for networks with correlated sources [10] and data storage for selective retrieval [11].

The rest of the paper is organized as follows. In Section II, we formally state the L -channel MD setup and briefly describe the approaches and regions of EC [1], ZB [2] and VKG [3]. To keep the notation simple, we first describe in Section III-A, the CMS scheme for the 3 descriptions scenario and extend it to the general case in Section III-B.

II. FORMAL DEFINITIONS AND PRIOR RESULTS

We first give a formal definition of the L -channel MD problem. We follow the notation in [3]. A source produces a sequence $X^n = (X^{(1)}, X^{(2)}, \dots, X^{(n)})$, which are n iid copies of a generic random variable X taking values in a finite alphabet \mathcal{X} . We denote $\mathcal{L} = \{1, \dots, L\}$. There are L encoding functions, $f_l(\cdot)$ $l \in \mathcal{L}$, which map X^n to the descriptions

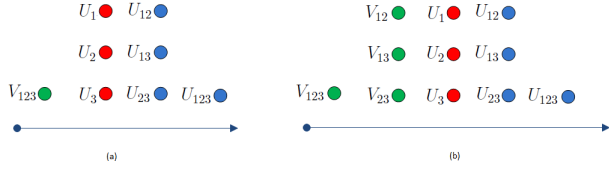


Figure 3. (a) Denotes the codebook generation order for VKG scheme. (b) represents the codebook generation for the proposed coding scheme.

distortion measure. Some of the new ideas introduced herein may have impact on these special cases, but such extensions are beyond the scope of this paper.

III. COMBINATORIAL MESSAGE SHARING

In this section, we describe the proposed encoding scheme which leads to a refined achievable rate region for the L -channel MD setup. To simplify notation and understanding, we first describe the scheme for the 3-descriptions case and offer intuitive arguments to show the achievability of the new region. We then extend the arguments to the L -channel case in Theorem 1.

A. 3-Descriptions scenario

The encoding order for the 3-descriptions VKG scheme is shown in Figure 3(a). Recall that the common codeword helps in coordinating the 3 descriptions. VKG employ *one* common codeword that is sent in all the 3 descriptions. However, when dealing with $L > 2$ descriptions, restricting to a single shared message might be suboptimal. The CMS scheme therefore will allow for ‘combinatorial message sharing’, i.e a common codeword may be sent in each (non-empty) subset of the descriptions.

The shared random variables are denoted by ‘ V ’. The base and the refinement layer random variables are denoted by ‘ U ’. For the 3 descriptions scenario, we have 11 auxiliary random variables which include 3 additional variables over the VKG scheme. These variables are denoted by V_{12} , V_{13} and V_{23} . The codeword corresponding to V_S is sent in *all* the descriptions $l \in \mathcal{S}$. For example, the codeword of V_{12} is sent in both the descriptions 1 and 2. This introduces a new layer in the encoding structure as shown in Figure 3 (b). This extra encoding layer can refine the control of redundancy across the messages, and hence lead to an improved rate-distortion region.

The codebook generation is done as follows. First, the codebook for V_{123} is generated containing $2^{nR''_{123}}$ independently generated codewords. Then codebooks for V_{12} , V_{13} and V_{23} (each containing $2^{nR''_{12}}$, $2^{nR''_{13}}$ and $2^{nR''_{23}}$ codewords respectively) are generated, conditioned on each codeword of V_{123} . Next, the base layer codebooks for U_l , $l \in \{1, 2, 3\}$ (each containing $2^{nR'_l}$ codewords) are generated conditioned on the codewords of all V_S such that $l \in \mathcal{S}$. For example, the codebooks for U_1 are generated conditioned on the codewords of V_{12} , V_{13} and V_{123} . Note that each codebook of U_1

contains $2^{nR'_1}$ codewords and there are such $2^{n(R''_{12}+R''_{13}+R''_{123})}$ codebooks.

The refinement layer codewords are generated similar to the VKG scheme. However, the codebook for $U_{\mathcal{K}}$, is now generated conditioned not only on the codewords of $\{U\}_{2^{\mathcal{K}-\phi-\mathcal{K}}}$ and $V_{\mathcal{L}}$, but also on the codewords of all V_S such that $|\mathcal{S} \cap \mathcal{K}| > 0$. For example, a single codeword for U_{12} is generated conditioned on each codeword tuple of $\{U_1, U_2, V_{12}, V_{13}, V_{23}, V_{123}\}$. Note that, there are $2^{n(R'_1+R'_2+R''_{12}+R''_{13}+R''_{123})}$ codewords in the codebook of U_{12} . Similarly codebooks for U_{13} , U_{23} and U_{123} are generated conditioned on codewords of $\{U_1, U_3, V_{12}, V_{13}, V_{23}, V_{123}\}$, $\{U_2, U_3, V_{12}, V_{13}, V_{23}, V_{123}\}$ and $\{U_1, U_2, U_3, V_{12}, V_{13}, V_{23}, V_{123}\}$ respectively.

The encoder, on observing X^n , tries to find a codeword from the codebook of V_{123} such that it is jointly typical with X^n . Using typicality arguments, it is easy to show that the probability of not finding such a codeword approaches zero if $R''_{123} \geq I(X; V_{123})$. Let us denote the selected codeword of V_{123} by v_{123} . The encoder next looks at the codebooks of V_{12} , V_{13} and V_{23} , which were generated conditioned on v_{123} , to find a triplet of codewords which are jointly typical with (X^n, v_{123}) . It can be shown using arguments similar to [3], [9], that the probability of not finding such a triplet approaches zero if the following conditions are satisfied $\forall \mathcal{Q} \subseteq \{12, 13, 23\}$:

$$\sum_{\mathcal{K} \in \mathcal{Q}} R''_{\mathcal{K}} \geq \sum_{\mathcal{K} \in \mathcal{Q}} H(V_{\mathcal{K}}|V_{123}) - H(\{V\}_{\mathcal{Q}}|V_{123}, X) \quad (3)$$

Let us denote these codewords by v_{12} , v_{13} and v_{23} . The encoders next step is to find an index tuple (i_1, i_2, i_3) such that $(U_1(i_1), U_2(i_2), U_3(i_3), U_{12}(i_1, i_2), U_{13}(i_1, i_3), U_{23}(i_2, i_3), U_{123}(i_1, i_2, i_3))$ is jointly typical with $(X^n, v_{123}, v_{12}, v_{13}, v_{23})$. Here $U_1(i_1)$ denotes the i_1 th codeword from the codebook of U_1 and $U_{12}(i_1, i_2)$ denotes the codeword of U_{12} generated conditioned on $(v_{123}, v_{12}, v_{13}, v_{23}, U_1(i_1), U_2(i_2))$. Similar notation is used for $U_2(i_2), U_3(i_3), U_{13}(i_1, i_3)$ etc. Again using similar arguments, we can show that the probability of not finding such an index tuple approaches zero if $\forall \mathcal{S} \subseteq \{1, 2, 3\}$:

$$\sum_{l \in \mathcal{S}} R'_l \geq \sum_{\mathcal{K} \in 2^{\mathcal{S}-\phi}} H(U_{\mathcal{K}}|\{U\}_{2^{\mathcal{K}-\phi-\mathcal{K}}}, \{V\}_{\mathcal{J}(\mathcal{K})}) - H(\{U\}_{2^{\mathcal{S}-\phi}}|V_{12}, V_{13}, V_{23}, V_{123}, X) \quad (4)$$

where $\mathcal{J}(\mathcal{K}) = \{\mathcal{S} : \mathcal{S} \in \{12, 13, 23, 123\}, |\mathcal{K} \cap \mathcal{S}| > 0\}$. We denote the codewords corresponding to this index tuple by $(u_1, u_2, u_3, u_{12}, u_{13}, u_{23}, u_{123})$.

The encoder sends the index of the base layer codewords in the corresponding descriptions. It also sends the index of codewords corresponding to V_S in *all* the descriptions $l \in \mathcal{S}$. For example, in description 1, the encoder sends the indices corresponding to v_{123}, v_{12}, v_{13} and u_1 . Similarly, descriptions 2 and 3 carry indices corresponding to $(v_{123}, v_{12}, v_{23}, u_2)$ and $(v_{123}, v_{13}, v_{23}, u_3)$ respectively. Therefore, rate for description l is $R_l = R'_l + \sum_{\mathcal{K} \in \mathcal{J}(l)} R''_{\mathcal{K}}$. Conditions on R_l can be obtained by substituting bounds from (3) and (4).

are denoted by $v_S^n(\{j\}_{\mathcal{I}_{W+(S)}}, j_S)$ $j_S \in \{1 \dots 2^{nR_S''}\}$. Note that to generate the codebooks for $V_S \forall S \in \mathcal{I}_W$, we need the codebooks of $V_S \forall S \in \mathcal{I}_{W+(Q)}$. The codebook generation follows the order indicated in figure 4.

Once all the codebooks of shared random variables are generated, the codebooks for the base layer random variables are generated. For each codeword tuple of $\{V\}_{\mathcal{I}_{1+(l)}}$, $2^{nR_l'}$ codewords of U_l are generated independently according to $\prod_{t=1}^n P_{U_l|\{V\}_{\mathcal{I}_{1+(l)}}}(u_l^{(t)}|\{v\}_{\mathcal{I}_{1+(l)}}^{(t)})$ and are denoted by $u_l^n(\{j\}_{\mathcal{I}_{1+(l)}}, i_l)$ $i_l \in \{1 \dots 2^{nR_l'}\}$. Then the codebooks for the refinement layers are formed by assigning a codeword $u_S^n(\{j\}_{\mathcal{J}(S)}, \{i\}_{\mathcal{S}})$ to each $S \in 2^{\mathcal{L}} - \{(1), (2) \dots (L)\}$ and $\forall \{j\}_{\mathcal{J}(S)}, \{i\}_{\mathcal{S}}$. These codewords are generated according to $\prod_{t=1}^n P_{U_S|\{V\}_{\mathcal{J}(S)}, \{U\}_{2^{\mathcal{S}}-S}}(u_S^{(t)}|\{v\}_{\mathcal{J}(S)}^{(t)}, \{u\}_{2^{\mathcal{S}}-S}^{(t)})$.

The encoder, on observing a sequence X^n attempts to find a set of codewords, one for each variable, such that they are all jointly typical. If the encoder succeeds in finding such a set, from the typical average lemma [9], it follows that the average distortions are less than $D_S, \forall S \in 2^{\mathcal{L}} - \phi$. However, if the encoder fails to find such a set, the average distortions are upper bounded by d_{max} (as the distortion measures are assumed to be bounded). Hence, if the probability of finding a set of jointly typical codewords approaches 1 the distortion conditions, (12), are met. We next show that, this probability approaches 1 if the rates satisfy (11).

Probability of error analysis : Let E_{VU} be the event of not finding a set of jointly typical codewords. We have,

$$P(E_{VU}) = P(E_V) + P(E_{VU}|E_V^c)P(E_V^c) \quad (13)$$

where E_V denotes the event of not finding a set of ‘shared codewords’ which are jointly typical with X^n .

Conditions on R_S'' : We can rewrite $P(E_V)$ as,

$$P(E_V) = P(E_{V,L}) + P(E_{V,L-1}|E_{V,L}^c)P(E_{V,L}^c) + \dots + P(E_{V,2}|E_{V,3}^c)P(E_{V,3}^c) \quad (14)$$

where $E_{V,W}$ denotes the event of not finding a set of codewords of $\{V\}_{(\mathcal{I}_W, \mathcal{I}_{W+})}$ which are jointly typical with X^n . It can be shown that, $P(E_{V,W}|E_{V,W+1}^c)$ can be made arbitrarily small if $\sum_{\mathcal{K} \in \mathcal{Q}} R_{\mathcal{K}}'' > \alpha_W(\mathcal{Q}) \forall \mathcal{Q} \subseteq \mathcal{I}_W$.

Conditions on R_S' : Similar arguments as in [3] can be used to show that, $P(E_{VU}|E_V^c)$ can be made arbitrarily small if $\sum_{l \in \mathcal{S}} R_l' > \beta(\mathcal{S}) \forall \mathcal{S} \subseteq \mathcal{L}$.

Conditions on R_S : Recall that the encoder sends the codewords of V_S (at rate R_S'') in all the descriptions $l \in \mathcal{S}$. It also sends the codewords of U_l (at rate R_l') in description l . Therefore the rate of description l can be bounded by:

$$R_l \geq R_l' + \sum_{\mathcal{K} \in \mathcal{J}(l)} R_{\mathcal{K}}'' \quad (15)$$

Combining (13), (14) and (15), proves the theorem. ■

Note that, the bounds on R_l in (15) are implicit in terms of the bounds on R_l' and $R_{\mathcal{K}}''$. Expressing them directly in terms information theoretic quantities is considerably harder and will be considered as part of future work.

Corollary 1. *The achievable rate-distortion region of Theorem 1 subsumes the VKG region, i.e. $\mathcal{RD}_{VKG} \subseteq \mathcal{RD}_{CMS}$*

Proof: The proof follows directly by setting $V_S = \Phi \forall S$ such that $|\mathcal{S}| < L$. We then have $\alpha_W(\mathcal{Q}) = 0 \forall \mathcal{Q}, W < L$. Substituting in (11), we get

$$R(\mathcal{S}) \geq \beta(\mathcal{S}) + |\mathcal{S}| \alpha_L(\mathcal{L}) \quad (16)$$

which is same as (2). ■

IV. COMMENT ON STRICT ENLARGEMENT

At the time of the submission of this manuscript, we did not know whether the CMS scheme leads to a strictly larger rate-distortion region compared to the VKG scheme, i.e., $\mathcal{RD}_{VKG} \subset \mathcal{RD}_{CMS}$. We recently proved that, this is indeed true, i.e., a combinatorial number of shared codewords, one for every subset of descriptions, can be used to achieve points outside the VKG region. The proof is available on arXiv [12] for interested readers.

V. CONCLUSION

In this paper, we proposed a new encoding scheme for the L -channel multiple descriptions problem. This results in a new achievable rate-distortion region which subsumes the most famous achievable region for this problem. The proposed encoding adds controlled redundancy by including a common codeword in every subset of the descriptions. The possible impact of the new ideas presented here, on practical multiple descriptions encoder design and on certain special cases of the setting will be studied as part of future work.

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