On the Role of Common Codewords in Quadratic Gaussian Multiple Descriptions Coding

Kumar Viswanatha, Emrah Akyol and Kenneth Rose
ECE Department, University of California - Santa Barbara
{kumar,eakyol,rose}@ece.ucsb.edu

***Abstract*—This paper focuses on the problem of $L-$channel quadratic Gaussian multiple description (MD) coding. We recently introduced a new encoding scheme in [1] for general $L-$channel MD problem, based on a technique called 'Combinatorial Message Sharing' (CMS), where every subset of the descriptions shares a distinct common message. The new achievable region subsumes the most well known region for the general problem, due to Venkataramani, Kramer and Goyal (VKG) [2]. Moreover, we showed in [3] that the new scheme provides a strict improvement of the achievable region for any source and distortion measures for which some 2-description subset is such that the Zhang and Berger (ZB) scheme achieves points outside the El-Gamal and Cover (EC) region. In this paper, we show a more surprising result: CMS outperforms VKG for a general class of sources and distortion measures, which includes scenarios where for all 2-description subsets, the ZB and EC regions coincide. In particular, we show that CMS strictly extends VKG region, for the L-channel quadratic Gaussian MD problem for all $L \geq 3$, despite the fact that the EC region is complete for the corresponding 2-descriptions problem. Using the encoding principles derived, we show that the CMS scheme achieves the complete rate-distortion region for several asymmetric cross-sections of the $L-$channel quadratic Gaussian MD problem, which have not been considered earlier.**

***Index Terms*—Multiple description coding, Combinatorial message sharing, Quadratic Gaussian multiple descriptions**

I. Introduction

The multiple descriptions (MD) problem has been studied extensively, yielding a series of advances, ranging from achievability [4], [5], [2], [6], [1], [3], [7] to converse results [8], [9], [10]. In the general MD setup, the encoder generates L-descriptions of the source for transmission over L available channels and it is assumed that the decoder receives a subset of the descriptions perfectly and the remaining are lost. The objective is to quantify the set of all achievable rate-distortion (RD) tuples for the $L-$rates $(R_1, \ldots, R_L)$ and distortion levels corresponding to the $2^L - 1$ possible description loss patterns $(D_{\mathcal{K}}, \mathcal{K} \subseteq \{1, \ldots, L\})$. One of the first achievable regions for the 2-channel MD problem was derived by El-Gamal and Cover (EC) in 1982 [4]. It was shown by Ozarow in [8] that the EC region is complete when the source is Gaussian and the distortion measure is mean squared error (MSE). Zhang and Berger (ZB), however, later showed in [5] that the EC coding scheme is strictly sub-optimal in general. In particular, for a binary source under Hamming distortion, sending a common codeword within the two descriptions can achieve points that are strictly outside the the EC region. The converse to the ZB scheme in still not known for general sources and distortion measures.

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Since then several researchers have worked on extending the EC and ZB approaches to the $L-$channel MD problem [2], [6], [9], [10]. An achievable scheme, due to Venkataramani, Kramer and Goyal (VKG) [2], directly builds on EC and ZB, and introduces a combinatorial number of refinement codebooks, one for each subset of the descriptions. Motivated by ZB, a *single* common codeword is also shared by all the descriptions, which assists in better coordination of the messages, improving the RD trade-off. We recently introduced a new coding scheme called 'Combinatorial Message Sharing' (CMS) in [1], wherein a distinct common codeword is shared by members of each subset of the transmitted descriptions. The new achievable RD region subsumes the VKG region for general sources and distortion measures. Moreover, we demonstrated in [3] that CMS achieves a strictly larger region than VKG for all $L > 2$, if there exists a 2-description subset for which ZB achieves points strictly outside the EC region. In particular, CMS achieves strict improvement for a binary source under Hamming distortion.

Ozarow's converse result [8] motivated researchers to seek extended results for the $L-$channel quadratic Gaussian MD problem [9], [10]. It was shown in [9] that a special case of the VKG coding scheme, called the 'correlated quantization' scheme (a generalization of Ozarow's encoding mechanism to $L-$channels), where *no common codewords are sent,* achieves the complete rate region, when only the individual and the central distortion constraints are imposed. A different and important line of attack focused on a practically interesting cross-section of the general MD problem, called the 'symmetric MD problem' (see [6]), based on encoding principles derived from Slepian and Wolf's random binning techniques. In fact, CMS principles can be extended to incorporate such random binning techniques, to utilize the underlying symmetry in the problem setup as illustrated recently in [7]. However, in this paper, we restrict ourselves to the general asymmetric setup to demonstrate the potential gains of using the common codewords of CMS for the quadratic Gaussian MD problem.

Optimality of EC for the 2-descriptions setup has led to a natural conjecture that common codewords do not play a necessary role in quadratic Gaussian MD coding, and all the achievable regions characterized so far neglect the common

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layer codewords (see eg., [2], [9], [10]). In this paper, we show that, surprisingly CMS strictly outperforms VKG for a Gaussian source under MSE distortion. More generally, we show that strict improvement holds for a general class of sources and distortion measures, which includes several scenarios in which, for every 2-description subset, ZB and EC lead to the same achievable region. We also show that the common codewords of CMS play a critical role in achieving the complete RD region for several asymmetric cross-sections of the $L-$channel quadratic Gaussian MD problem.

We note that, due to severe space constraints, in this paper, we avoid restating some of the prior results related to MD and refer to [4], [5], [2] for the details of EC, ZB and VKG scheme, respectively. In the following section, we begin with the formal definitions and a brief description of CMS.

II. Formal definitions and the CMS coding scheme

A source produces a sequence of n iid random variables, denoted by $X^n = \left(X^{(1)}, X^{(2)} \ldots, X^{(n)}\right)$, of a generic random variable X taking values in a finite alphabet $\mathcal{X}$. We denote $\mathcal{L} = \{1, \ldots, L\}$. There are L encoding functions, $f_l(\cdot)$ $l \in \mathcal{L}$, which map X^n to the descriptions $J_l = f_l(X^n)$, where $J_l \in \{1, \ldots B_l\}$ for some $B_l > 0$. The rate of description l is defined as $R_l = \log_2(B_l)$. Each of the descriptions are sent over a separate channel and are either received at the decoder error free or are completely lost. There are $2^L - 1$ decoding functions for each possible received combination of the descriptions $\hat{X}^n_{\mathcal{K}} = \left(\hat{X}^{(1)}_{\mathcal{K}}, \ldots, \hat{X}^{(n)}_{\mathcal{K}}\right) = g_{\mathcal{K}}(J_l : l \in \mathcal{K})$, $\forall \mathcal{K} \subseteq \mathcal{L}, \mathcal{K} \neq \phi$, where $\hat{X}_{\mathcal{K}}$ takes on values on a finite set $\hat{\mathcal{X}}_{\mathcal{K}}$, and ϕ denotes the null set. When a subset $\mathcal{K}$ of the descriptions are received at the decoder, the distortion is measured as $D_{\mathcal{K}} = E\left[\frac{1}{N}\sum_{t=1}^{n} d_{\mathcal{K}}(X^{(t)}, \hat{X}^{(t)}_{\mathcal{K}})\right]$ for some bounded distortion measures $d_{\mathcal{K}}(\cdot)$ defined as $d_{\mathcal{K}} : \mathcal{X} \times \hat{\mathcal{X}}_{\mathcal{K}} \to \mathcal{R}$. We say that a rate-distortion tuple $(R_i, D_{\mathcal{K}} : i \in \mathcal{L}, \mathcal{K} \subseteq \mathcal{L}, \mathcal{K} \neq \phi)$ is achievable if there exit L encoding functions with rates $(R_1 \ldots, R_L)$ and $2^L - 1$ decoding functions yielding distortions $D_{\mathcal{K}}$. The closure of the set of all achievable rate-distortion tuples is defined as the '*L-channel multiple descriptions RD region*'. Note that, this region has $L + 2^L - 1$ dimensions.

In what follows, $2^{\mathcal{S}}$ denotes the set of all subsets (power set) of any set $\mathcal{S}$ and $|\mathcal{S}|$ denotes the set cardinality. Note that $|2^{\mathcal{S}}| = 2^{|\mathcal{S}|}$. $\mathcal{S}^c$ denotes the set complement. For two sets $\mathcal{S}_1$ and $\mathcal{S}_2$, we denote the set difference by $\mathcal{S}_1 - \mathcal{S}_2 = \{\mathcal{K} : \mathcal{K} \in \mathcal{S}_1, \mathcal{K} \notin \mathcal{S}_2\}$. We use the shorthand $\{U\}_{\mathcal{S}}$ for $\{U_{\mathcal{K}} : \mathcal{K} \in \mathcal{S}\}$[1]. Before describing CMS, we define the following subsets of $2^{\mathcal{L}}$:

$$\begin{aligned} \mathcal{I}_W &= \{\mathcal{S} : \mathcal{S} \in 2^{\mathcal{L}},\ |\mathcal{S}| = W\} \\ \mathcal{I}_{W+} &= \{\mathcal{S} : \mathcal{S} \in 2^{\mathcal{L}},\ |\mathcal{S}| > W\} \end{aligned} \tag{1}$$

Let $\mathcal{B}$ be any non-empty subset of $\mathcal{L}$ with $|\mathcal{B}| \leq W$. We define the following subsets of $\mathcal{I}_W$ and $\mathcal{I}_{W+}$:

$$\begin{aligned} \mathcal{I}_W(\mathcal{B}) &= \{\mathcal{S} : \mathcal{S} \in \mathcal{I}_W,\ \mathcal{B} \subseteq \mathcal{S}\} \\ \mathcal{I}_{W+}(\mathcal{B}) &= \{\mathcal{S} : \mathcal{S} \in \mathcal{I}_{W+},\ \mathcal{B} \subseteq \mathcal{S}\} \end{aligned} \tag{2}$$

We also define $\mathcal{J}(\mathcal{K}) = \bigcup_{l \in \mathcal{K}} \mathcal{I}_{1+}(l)$.

Next, we briefly describe the CMS encoding scheme in [1]. Recall that, unlike VKG, CMS allows for 'combinatorial message sharing', i.e a common codeword is sent in each (non-empty) subset of the descriptions. The shared random variables are denoted by 'V'. The base and the refinement layer random variables are denoted by 'U'. Suppose we are given $P\left(\{v\}_{2^{\mathcal{L}} - \{(1),(2),\ldots(L)\}}, \{u\}_{2^{\mathcal{L}}} | x\right)$ and functions $\psi_{\mathcal{S}}(\cdot)$ satisfying the following conditions:

$$D_{\mathcal{S}} \geq E\left[d_{\mathcal{S}}\left(X, \psi_{\mathcal{S}}\left(U_{\mathcal{S}}\right)\right)\right] \tag{3}$$

First, the codebook for $V_{\mathcal{L}}$ is generated. Then, the codebooks for $V_{\mathcal{S}}$, $|S| = W$ are generated in the order $W = L-1, L-2 \ldots 2$. $2^{nR^{''}_{\mathcal{Q}}}$ codewords of $V_{\mathcal{Q}}$ are independently generated conditioned on each codeword tuple of $\{V\}_{\mathcal{I}_{W+}(\mathcal{Q})}$. This is followed by the generation of the base layer codebooks, i.e. U_l, $l \in \mathcal{L}$. Conditioned on each codeword tuple of $\{V\}_{I_{1+}(l)}$, $2^{nR^{'}_l}$ codewords of U_l are generated independently. Then the codebooks for the refinement layers are formed by generating a single codeword for $U_{\mathcal{S}}$, $|\mathcal{S}| > 1$ conditioned on every codeword tuple of $(\{V\}_{\mathcal{J}(\mathcal{S})}, \{U\}_{2^{\mathcal{S}} - \mathcal{S}})$. Observe that the base and the refinement layers in the CMS scheme are similar to that in the VKG scheme, except that they are now generated conditioned on a subset of the shared codewords.

The encoder employs joint typicality encoding, i.e., on observing a typical sequence x^n, it tries to find a jointly typical codeword tuple, one from each codebook. As with VKG, the codeword index of U_l (at rate $R^{'}_l$) is sent in description l. However, now the codeword index of $V_{\mathcal{S}}$ (at rate $R^{''}_{\mathcal{S}}$) is sent in *all* the descriptions $l \in \mathcal{S}$. Therefore the rate of description l is:

$$R_l = R^{'}_l + \sum_{\mathcal{K} \in \mathcal{J}(l)} R^{''}_{\mathcal{K}} \tag{4}$$

The achievable region follows by finding conditions on the rates to ensure finding jointly typical codewords. Further, the condition (3) ensures that the distortion constraints are satisfied. We refer to [1] for a detailed proof and a single letter characterization of the achievable region.

III. Strict Improvement for a General Class of Sources and Distortion Measures

We begin by defining $\mathcal{Z}_{ZB}$, the set of all sources (for given distortion measures at the decoders), for which there exists an operating point $(R_1, R_2, D_1, D_2, D_{12})$ that *cannot* be achieved by an 'independent quantization' mechanism using the ZB coding scheme. More specifically, $X \in \mathcal{Z}_{ZB}$, if there exists a strict suboptimality in the ZB region when the closure is defined only over joint densities for the auxiliary random

[1]Note the difference between $\{U\}_{\mathcal{S}}$ and $U_{\mathcal{S}}$. $\{U\}_{\mathcal{S}}$ is a set of variables, whereas $U_{\mathcal{S}}$ is a single variable.

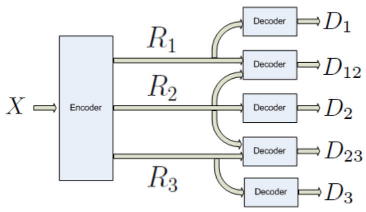

Figure 1. The cross-section that we consider in order to prove that CMS achieves points outside the VKG region for a general class of source and distortion measures. CMS achieves the the complete RD region for this setup for several distortion regimes for the quadratic Gaussian MD problem.

variables satisfying the following conditions:

$$\begin{aligned} P(U_1, U_2|X, V_{12}) &= P(U_1|X, V_{12})P(U_2|X, V_{12}) \\ E\left[d_{\mathcal{K}}(X, \psi_{\mathcal{K}}(U_{\mathcal{K}}))\right] &\leq D_{\mathcal{K}}, \quad \mathcal{K} \in \{1, 2, 12\} \\ U_{12} &= f(U_1, U_2, V_{12}) \end{aligned} \tag{5}$$

where f is any deterministic function. We will show in Theorem 1 that $\forall X \in \mathcal{Z}_{ZB}$, $\mathcal{RD}_{VKG} \subset \mathcal{RD}_{CMS}$.

Before stating the result we describe the particular cross-section of the RD region that we will use to prove strict improvement in Theorem 1. Consider a 3-descriptions MD setup for a source X wherein we impose constraints only on distortions $(D_1, D_2, D_3, D_{12}, D_{23})$ and set the rest of the distortions, (D_{13}, D_{123}) to ∞. This cross-section is schematically shown in Fig. 1. To illustrate the gains underlying CMS, here we restrict ourselves to the setting wherein we further impose $D_1 = D_3$ and $D_{12} = D_{23}$. The points in this cross-section, achievable by VKG and CMS, are denoted by $\overline{\mathcal{RD}}_{VKG}(X)$ and $\overline{\mathcal{RD}}_{CMS}(X)$, respectively. We note that the symmetric setting is considered *only* for simplicity. The arguments can be easily extended to the asymmetric framework.

This particular symmetric cross-section of the 3-descriptions MD problem is equivalent to the corresponding 2-descriptions problem, in the sense that, one could use any coding scheme to generate bit-streams for descriptions 1 and 2, respectively. Description 3 would then carry a replica (exact copy) of the bits sent in description 1. Due to the underlying symmetry in the problem setup, the distortion constraints at all the decoders are satisfied. Hence an achievable region based on the ZB coding scheme can be derived as follows. Let $(G_{12}, F_1, F_2, F_{12})$ be any random variables jointly distributed with X and taking values over arbitrary finite alphabets. Then the following RD-region is achievable for which there exist functions $(\psi_1, \psi_2, \psi_{12})$ such that $R_1 = R_3$, $D_1 = D_3$, $D_{12} = D_{23}$ and:

$$\begin{aligned} R_1 \geq I(X; F_1, G_{12}),\ R_2 \geq I(X; F_2, G_{12}) \\ R_1 + R_2 \geq 2I(X; G_{12}) + H(F_1|G_{12}) + H(F_2|G_{12}) \\ -H(F_1, F_2, F_{12}|X, G_{12}) + H(F_{12}|F_1, F_2, G_{12}) \\ D_{\mathcal{K}} \geq E\left[d_{\mathcal{K}}(X, \psi_{\mathcal{K}}(F_{\mathcal{K}}))\right],\ \mathcal{K} \in \{1, 2, 12\} \end{aligned} \tag{6}$$

The closure of achievable RD-tuples over all random variables $(G_{12}, M_1, M_2, M_{12})$ is denoted by $\overline{\mathcal{RD}}(X)$. In the following theorem, we will show that $\overline{\mathcal{RD}}(X) \subseteq \overline{\mathcal{RD}}_{CMS}(X)$. We also show that the VKG coding scheme *cannot* achieve the above RD region, i.e., $\overline{\mathcal{RD}}_{VKG}(X) \subset \overline{\mathcal{RD}}(X)$, if $X \in \mathcal{Z}_{ZB}$. We note that in Theorem 1, we focus only on the 3-descriptions setting. However, the results can be easily extended to the general $L-$descriptions scenario. Also note that $\overline{\mathcal{RD}}_{CMS}(X)$ could be strictly larger than $\overline{\mathcal{RD}}(X)$, in general.

Theorem 1. *(i) For the setup shown in Fig. 1 the CMS scheme achieves $\overline{\mathcal{RD}}(X)$, i.e., $\overline{\mathcal{RD}}(X) \subseteq \overline{\mathcal{RD}}_{CMS}(X)$.*

*(ii) If $X \in \mathcal{Z}_{ZB}$, then there exists points in $\overline{\mathcal{RD}}(X)$ that **cannot** be achieved by the VKG encoding scheme, i.e., $\overline{\mathcal{RD}}_{VKG} \subset \overline{\mathcal{RD}}(X)$,*

Remark 1. It directly follows from (i) and (ii) that $\mathcal{RD}_{VKG} \subset \mathcal{RD}_{CMS}$ for the $L-$channel MD problem $\forall L \geq 3$, if $X \in \mathcal{Z}_{ZB}$.

Proof. We first provide an intuitive argument to justify the claim and then follow it up with a formal argument. Due to the underlying symmetry in the setup CMS introduces common layer random variables $V_{123} = G_{12}$ and $V_{13} = F_1$. It then sends the codeword of V_{13} is both descriptions 1 and 3 (i.e., $U_1 = U_3 = V_{13}$). Hence it is sufficient for the encoder to generate enough codewords of $U_2 = F_2$ (conditioned on V_{123}) to maintain joint typicality with the codewords of $V_{13} = F_1$. However VKG is forced to set the common layer random variable V_{13} to a constant. Thus, in this case, the encoder needs to generate enough number of codewords of U_2 so as to maintain joint typicality individually with the codewords of U_1 and U_3, which are now generated independently conditioned on V_{123}, entailing some excess rate for U_2[2].

Part (i) of the theorem is straightforward to prove. We set $V_{123} = G_{12}$, $V_{13} = F_1$, $U_2 = F_2$, $U_{12} = U_{23} = F_{12}$ and $U_1 = U_3 = V_{13}$ and the rest of the random variables to constants in the CMS achievable region in [1]. This leads to the RD region in (6).

We next prove (ii). We consider one particular boundary point of (6) and show that this cannot be achieved by VKG. Let D_1, D_2 and D_{12} be fixed. Consider the following quantity:

$$\begin{aligned} R^*_{VKG}(D_1, D_2, D_{12}) = \inf \Big\{ R_2 : R_1 = R_3 = R_X(D_1) \\ (R_1, R_2, R_3, D_1, D_2, D_1, D_{12}, D_{12}) \in \overline{\mathcal{RD}}_{VKG}(X) \Big\} \end{aligned} \tag{7}$$

Note that the corresponding quantity achievable using $\overline{\mathcal{RD}}_{CMS}(X)$ is given by the solution to the following optimization problem:

$$\begin{aligned} R^*_{CMS}(D_1, D_2, D_{12}) = \inf \Big\{ I(U_2; X, U_1|V_{123}) \\ I(X; V_{123}) + I(U_{12}; X|V_{123}, U_1, U_2) \Big\} \end{aligned} \tag{8}$$

where the infimum is over all densities $P(V_{123}, U_1, U_2, U_{12}|X)$, where $P(V_{123}, U_1|X)$ is any

[2]It might be tempting to conclude that the suboptimality in VKG is due to conditions for joint typicality of all the codewords, while for this cross-section, joint typicality of codewords of U_1 and U_3 is unnecessary. However, it is possible to show that common codewords provide strict improvement even when joint typicality only within prescribed subsets is imposed. The details are omitted here.

density for which there exists a function $\psi_1(\cdot)$ such that:

$$I(X; V_{123}, U_1) = R(D_1), \qquad E\left[d_1(X, \psi_1(U_1))\right] = D_1 \quad (9)$$

We will show that $R^*_{VKG} > R^*_{CMS}$. We next specialize and restate $\overline{\mathcal{RD}}_{VKG}(X)$ for the considered cross-section. Let $(V_{123}, U_1, U_2, U_3, U_{12}, U_{23})$ be any random variables jointly distributed with X taking values on arbitrary alphabets. Then, $\overline{\mathcal{RD}}_{VKG}$ contains all rates and distortions for which there exist functions $\psi_1(\cdot), \psi_2(\cdot), \psi_3(\cdot), \psi_{12}(\cdot), \psi_{23}(\cdot)$, such that:

$$\begin{aligned} R_i &\geq I(X; U_i, V_{123}),\ i \in \{1,2,3\} \\ R_i + R_2 &\geq 2I(X; V_{123}) + I(U_i; U_2|V_{123}) \\ &\quad + I(X; U_i, U_2, U_{i2}|V_{123}),\ i \in \{1,3\} \\ R_1 + R_3 &\geq 2I(X; V_{123}) + H(U_1|V_{123}) \\ &\quad + H(U_3|V_{123}) - H(U_1, U_3|X, V_{123}) \\ R_1 + R_2 + R_3 &\geq 3I(X; V_{123}) + \sum_{i=1}^{3} H(U_i|V_{123}) \\ &\quad + \sum_{\mathcal{K} \in \{12,23\}} H(U_{\mathcal{K}}|\{U\}_{\mathcal{K}}, V_{123}) \\ &\quad - H(U_1, U_2, U_3, U_{12}, U_{23}|X, V_{123}) \end{aligned} \quad (10)$$

$$E\left(d_{\mathcal{K}}(X, \psi_{\mathcal{K}}(U_{\mathcal{K}}))\right) \leq D_{\mathcal{K}},\ \mathcal{K} \in \{1,2,3,12,13\} \quad (11)$$

where $R_1 = R_3$, $D_1 = D_3$ and $D_{12} = D_{23}$. Observe that the random variables U_{13} and U_{123} have been set to constants as we do not impose distortion constraints D_{13} and D_{123}, respectively. We can further restrict the conditional density $P(U_{12}, U_{23}|X, V_{123}, U_1, U_2, U_3)$ to be equal to:

$$P(U_{12}|X, V_{123}, U_1, U_2)P(U_{23}|X, V_{123}, U_2, U_3) \quad (12)$$

without any loss of optimality.

Next imposing $R_1 = R_3 = R_X(D_1)$ in (10), enforces the density $P(V_{123}, U_1, U_3|X)$ to satisfy the following constraints:

$$\begin{aligned} I(X; V_{123}, U_i) &= R(D_1),\ i \in \{1,3\} \\ E\left[d_i(X, \psi_i(V_{123}, U_i))\right] &= D_1,\ i \in \{1,3\} \\ P(U_1, U_3|X, V_{123}) &= P(U_1|X, V_{123}) \times P(U_3|X, V_{123}) \end{aligned} \quad (13)$$

where the last condition is required to satisfy the constraint on $R_1 + R_3$ in (10). Therefore, using (10) and (12) we have:

$$R^*_{VKG} = \inf \Big\{ I(X; V_{123}) + I(U_2; U_1, U_3, X|V_{123}) + I(X; U_{12}|U_1, U_2, V_{123}) + I(X; U_{23}|U_2, U_3, V_{123}) \Big\} \quad (14)$$

where the infimum is over all joint densities $P(V_{123}, U_1, U_2, U_3, U_{12}, U_{23}|X)$ satisfying (13) for which there exist functions $\psi_2(\cdot), \psi_{12}(\cdot), \psi_{23}(\cdot)$ satisfying the distortion constraints in (11).

From (14) and (8) it follows that R^*_{VKG} is equal to R^*_{CMS} if and only if the two quantities on the RHS of (14) and (8), respectively, are equal. However for any joint density, we have $I(U_2; U_1, U_3, X|V_{123}) \geq I(U_2; U_1, X|V_{123})$ and $I(X; U_{23}|V_{123}, U_2, U_3) \geq 0$. Also note that the constraints in (9) are a subset of the constraints in (13). Hence for R^*_{VKG} to be equal to R^*_{CMS}, any joint density which achieves R^*_{VKG} must satisfy the following conditions:

(i) The joint density of $(X, V_{123}, U_1, U_2, U_{12})$ must be the same as the corresponding joint density which achieves R^*_{CMS} (in (8)).

(ii)$I(U_2; U_3|V_{123}, U_1, X) = 0$, $I(X; U_{23}|V_{123}, U_2, U_3) = 0$. The constraint $I(X; U_{23}|V_{123}, U_2, U_3) = 0$ implies that X and U_{23} are independent given V_{123}, U_2 and U_3. Equivalently this constraint implies that the reconstruction $\hat{X}_{23}$ can be written as a deterministic function of V_{123}, U_2 and U_3, i.e., for R^*_{VKG} to be equal to R^*_{CMS}, there must exist a function $\psi_{23}(V_{123}, U_2, U_3)$ such that $E\left(d_{23}(X, \tilde{\psi}_{23}(V_{123}, U_2, U_3))\right) \leq D_{23} = D_{12}$. On the other hand, the constraint $I(U_2; U_3|V_{123}, U_1, X) = 0$ implies that $H(U_3|V_{123}, U_1, X) = H(U_3|V_{123}, U_1, U_2, X)$. However, the joint density of (X, V_{123}, U_1, U_3) must satisfy (13) for $R_1 = R_3 = R_X(D_1)$ to hold, i.e., $H(U_3|V_{123}, U_1, X) = H(U_3|V_{123}, X)$. Hence for R^*_{VKG} to be equal to R^*_{CMS}, we require:

$$H(U_3|V_{123}, X) = H(U_3|V_{123}, U_1, U_2, X) \quad (15)$$

which implies that $U_2 \leftrightarrow (X, V_{123}) \leftrightarrow U_3$ must hold. Recall that the density $P(U_3, V_{123}|X)$ is RD-optimal at D_1 and the joint density $P(U_2, V_{123}|X, U_1)$ must be identical to the joint density which achieves R^*_{CMS} (from condition (i)). Hence, it follows that, if $X \in \mathcal{Z}_{ZB}$, there exists at least one RD tuple in $\overline{\mathcal{RD}}(X)$ that cannot be achieved if we constrain the joint density to simultaneously satisfy both the conditions (i) and (ii), proving the theorem. □

Discussion: A direct consequence of the above theorem is that, if $X \in \mathcal{Z}_{ZB}$, then the common layer codewords of CMS provide strict improvement in the achievable region as compared to not using them, i.e., if $X \in \mathcal{Z}_{ZB}$, $\mathcal{RD}_{VKG}\Big|_{V_{\mathcal{L}}=\Phi} \subseteq \mathcal{RD}_{VKG} \subset \mathcal{RD}_{CMS}$, where $\mathcal{RD}\Big|_{V_{\mathcal{L}}=\Phi}$ denotes the VKG region when the common layer random variable (denoted by $V_{\mathcal{L}}$) is set to a constant Φ[3]. In fact, it is possible to show that, whenever $X \in \mathcal{Z}_{EC}$, $\mathcal{RD}\Big|_{V_{\mathcal{L}}=\Phi} \subset \mathcal{RD}_{CMS}$, where $\mathcal{Z}_{EC}$ is defined as the set of all sources for which there exists an operating point (with respect to the given distortion measures) that *cannot* be achieved by an 'independent quantization' mechanism using the EC coding scheme, i.e., if there exists an operating point that *cannot* be achieved by EC using a joint density for the auxiliary random variables satisfying:

$$\begin{aligned} P(U_1, U_2|X) &= P(U_1|X)P(U_2|X) \\ E\left[d_{\mathcal{K}}(X, \psi_{\mathcal{K}}(U_{\mathcal{K}}))\right] &\leq D_{\mathcal{K}},\ \mathcal{K} \in \{1,2,12\} \\ U_{12} &= f(U_1, U_2) \end{aligned} \quad (16)$$

where f is any deterministic function. Note that the set $\mathcal{Z}_{ZB}$ is a subset of $\mathcal{Z}_{EC}$. Also observe that if $X \notin \mathcal{Z}_{EC}$, the concatenation of two independent optimal quantizers is optimal in

[3]Note that setting $V_{\mathcal{L}}$ to a constant in VKG is equivalent to setting all the common layer random variables to constants in CMS.

achieving a joint reconstruction. While this condition could be satisfied for specific values of D_1, D_2 and D_{12}, it is seldom achieved *for all* values of (D_1, D_2, D_{12}). Though such sources are of some theoretical interest, the multiple descriptions encoding for such sources is degenerate. Hence with some trivial exceptions, it can be asserted that the common layer codewords in CMS can be used to achieve a strictly larger region (compared to not using any common codewords) for all sources and distortion measures, $\forall L \geq 3$.

IV. Gaussian MSE Setting

In the following theorem we show that, under MSE, a Gaussian source belongs to $\mathcal{Z}_{ZB}$.

Theorem 2. *(i) CMS achieves the* ***complete*** *RD region for the symmetric 3-descriptions quadratic Gaussian setup shown in Fig. 1.*

(ii) The VKG encoding scheme cannot achieve all the points in the region, i.e., $\overline{\mathcal{RD}}_{VKG} \subset \overline{\mathcal{RD}}_{CMS}$.

Remark 2. It follows from (i) and (ii) that $\mathcal{RD}_{VKG} \subset \mathcal{RD}_{CMS}$ for the $L-$channel quadratic Gaussian MD problem $\forall L > 2$.

Proof. Proof of (i) is straightforward and follows directly from the proof of Theorem 1. Hence, we only prove (ii). Specifically, we show that, a Gaussian random variable, under MSE, belongs to $\mathcal{Z}_{ZB}$. (ii) then follows directly from Theorem 1. Due to space constraints, in this paper, we only provide a high level intuition for the proof. However, we refer the readers to a longer version of this paper [11] for a detailed proof.

Consider the 2-description quadratic Gaussian problem. It follows from Ozarow's results (see also [4]) that, if $D_{12} \leq D_1 + D_2 - 1$, then the following rate region is achievable (and complete):

$$R_{\mathcal{K}} \quad \geq \quad \frac{1}{2}\log\frac{1}{D_{\mathcal{K}}}, \ \mathcal{K} \in \{1, 2, 12\} \tag{17}$$

i.e., there is no excess rate incurred due to encoding the source using two descriptions. Observe that the excess sum rate term in the ZB region must be set to zero to achieve the above rate-region. It is possible to show that, if we restrict the optimization to conditionally independent joint densities, then it is impossible to simultaneously satisfy all the distortions and achieve $I(U_1; U_2|V_{12}) = 0$, which is necessary to set the excess sum rate term in the ZB region to zero. □

Note that, as $\mathcal{Z}_{ZB} \subseteq \mathcal{Z}_{EC}$, a Gaussian source under MSE belongs to $\mathcal{Z}_{EC}$. Hence, the 'correlated quantization' scheme (an extreme special case of VKG) which has been proven to be complete for several cross-sections of the $L-$descriptions quadratic Gaussian MD problem [9], is strictly suboptimal in general.

V. Points on the Boundary

An important question that has not been addressed thus far is whether CMS achieves the complete rate region for other cross-sections of the $L-$channel quadratic Gaussian MD setup. In fact, it is possible to show that CMS achieves the complete region for several asymmetric cross-sections of the setup shown in Fig. 1. In particular, if $D_1 \leq D_3$, then CMS achieves the complete rate region if:

$$D_{23} = \frac{\sigma_2^2\sigma_3^2\left(1-\rho^2\right)}{\sigma_2^2\sigma_3^2\left(1-\rho^2\right)+\sigma_2^2+\sigma_3^2-2\sigma_2\sigma_3\rho} \tag{18}$$

where $\sigma_k^2 = \frac{D_k}{1-D_k}$ $k \in \{1, 2, 3\}$ and $\rho = \rho_{23}^*\frac{\sigma_2}{\sigma_3}$ where ρ_{23}^* is the optimum correlation required to achieve D_{23} (see eg. [12]). The optimal coding scheme resembles that in Theorem 1, but the information carried in descriptions 1 and 3 are not identical. Here, description 1 carries refinement information required to achieve $D_1 \leq D_3$, along with all the bits transmitted in description 3. A corresponding result can be derived if $D_3 \leq D_1$, but we omit restating the conditions here.We also note that these results can also be extended to the $L-$channel setup in an analogous way. We refer the readers to a longer version of the paper [11] for the details of the proof.

VI. Conclusion

In this paper, we showed that CMS achieves a strictly larger region compared to VKG for a general class of sources and distortion measures, which includes the quintessential setting of Gaussian source under mean squared error. As a consequence, it follows that the 'correlated quantization' scheme (an extreme special case of VKG), is strictly suboptimal in general. We also showed that CMS achieves the complete rate region for several asymmetric cross-sections of the 3-description setup shown in Fig. 1.

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