

# The Lossy Common Information of Correlated Sources

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**Abstract**—The two most prevalent notions of common information (CI) are due to Wyner and Gács–Körner and both the notions can be stated as two different characteristic points in the lossless Gray–Wyner region. Although the information theoretic characterizations for these two CI quantities can be easily evaluated for random variables with infinite entropy (e.g., continuous random variables), their operational significance is applicable only to the lossless framework. The primary objective of this paper is to generalize these two CI notions to the lossy Gray–Wyner network, which hence extends the theoretical foundation to general sources and distortion measures. We begin by deriving a single letter characterization for the lossy generalization of Wyner’s CI, defined as the minimum rate on the shared branch of the Gray–Wyner network, maintaining minimum sum transmit rate when the two decoders reconstruct the sources subject to individual distortion constraints. To demonstrate its use, we compute the CI of bivariate Gaussian random variables for the entire regime of distortions. We then similarly generalize Gács and Körner’s definition to the lossy framework. The latter half of this paper focuses on studying the tradeoff between the total transmit rate and receive rate in the Gray–Wyner network. We show that this tradeoff yields a contour of points on the surface of the Gray–Wyner region, which passes through both the Wyner and Gács–Körner operating points, and thereby provides a unified framework to understand the different notions of CI. We further show that this tradeoff generalizes the two notions of CI to the excess sum transmit rate and receive rate regimes, respectively.

**Index Terms**—Common information, Gray-Wyner network, multiterminal source coding.

## I. INTRODUCTION

THE quest for a meaningful and useful notion of common information (CI) of two discrete random variables (denoted by  $X$  and  $Y$ ) has been actively pursued by

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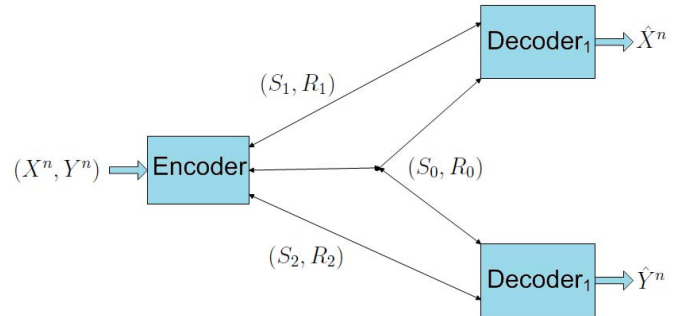


Fig. 1. The Gray-Wyner network.

researchers in information theory for over three decades. A seminal approach to quantify CI is due to Gács and Körner [1] (denoted here by  $C_{GK}(X, Y)$ ), who defined it as the maximum amount of information relevant to both random variables, one can extract from the knowledge of either one of them. Their result was of considerable theoretical interest, but also fundamentally negative in nature. They showed that  $C_{GK}(X, Y)$  is usually much smaller than the mutual information and is non-zero only when the joint distribution satisfies certain unique properties. Wyner proposed an alternative notion of CI [2] (denoted here by  $C_W(X, Y)$ ) inspired by earlier work in multi-terminal source coding [3]. Wyner’s CI is defined as:

$$C_W(X, Y) = \inf I(X, Y; U) \quad (1)$$

where the infimum is over all random variables,  $U$ , such that  $X \leftrightarrow U \leftrightarrow Y$  form a Markov chain in that order. He showed that  $C_W(X, Y)$  is equal to the minimum rate on the shared branch of the lossless Gray-Wyner network (described in section II-A and Fig. 1), when the sum rate is constrained to be the joint entropy. In other words, it is the minimum amount of shared information that must be sent to both decoders, while restricting the overall transmission rate to the minimum,  $H(X, Y)$ .

We note that although  $C_{GK}(X, Y)$  and  $C_W(X, Y)$  were defined from theoretical standpoints, they play important roles in understanding the performance limits in several practical networking and database applications, see eg., [4]. We further note in passing that several other definitions of CI, with applications in different fields, have appeared in the literature [5] and [6], but are less relevant to us here.

Although the quantity in (1) can be evaluated for random variables with infinite entropy (eg. continuous random variables), for such random variables it lacks the underlying theoretical interpretation, namely, as a distinctive operating point

in the Gray-Wyner region, and thereby mismatches Wyner's original reasoning. This largely compromises its practical significance and calls for a useful generalization which can be easily extended to infinite entropy distributions. Our primary step is to characterize a lossy coding extension of Wyner's CI (denoted by  $C_W(X, Y; D_1, D_2)$ ), defined as the minimum rate on the shared branch of the Gray-Wyner network at minimum sum rate when the sources are decoded at respective distortions of  $D_1$  and  $D_2$ . Note that the minimum sum rate at distortions  $D_1$  and  $D_2$  is given by Shannon's rate distortion function, hereafter denoted by  $R_{X,Y}(D_1, D_2)$ . In this paper, our main objective is to derive an information theoretic characterization for  $C_W(X, Y; D_1, D_2)$  for general sources and distortion measures. Using this characterization, we derive the lossy CI of two correlated Gaussian random variables for the entire regime of distortions. This example highlights several important characteristics of the lossy CI and challenges that underlie optimal encoding in the Gray-Wyner network. We note that although there is no prior work on characterizing  $C_W(X, Y; D_1, D_2)$ , in a recent work [7], Xu et al. provided an asymptotic definition for  $C_W(X, Y; D_1, D_2)$ <sup>1</sup> and showed that there exists a region of small enough distortions where  $C_W(X, Y; D_1, D_2)$  coincides with Wyner's single letter characterization in (1). We further note that there have been other physical interpretations of both notions of CI, irrespective of the Gray-Wyner network, including already in [1] and [2], whose connections with the lossy generalizations we consider herein are less direct and beyond the scope of this paper.

The last section of the paper focuses on the tradeoff between the total transmit rate and receive rate in the Gray-Wyner network, which directly relates the two notions of CI. Although it is well known that the two definitions of CI can be characterized as two extreme points in the Gray-Wyner region, no contour with operational significance is known which connects them. We show that the tradeoff between transmit and receive rates leads to a contour of points on the boundary of the Gray-Wyner region, which passes through the operating points of both Wyner and Gács-Körner. Hence, this tradeoff plays an important role in gaining theoretical insight into more general notions of shared information. Beyond theoretical insight, this tradeoff also plays a role in understanding fundamental limits in many practical applications including storage of correlated sources and minimum cost routing for networks (see eg., [4]). Motivated by these applications, we consider the problem of deriving a single letter characterization for the optimal tradeoff between the total transmit versus receive rate in the Gray-Wyner network. We provide a complete single letter characterization for the lossless setting. We develop further insight into this tradeoff by defining two quantities  $C(X, Y; R')$  and  $K(X, Y; R'')$ , which quantify the shared rate as a function of the total transmit and receive rates, respectively. These two quantities generalize the lossless notions of CI to the excess sum transmit rate and receive rate regimes, respectively. Finally, we use these properties to derive alternate characterizations for the two definitions of CI under a broader unified framework. We note that a preliminary version

<sup>1</sup>They denote  $C_W(X, Y; D_1, D_2)$  by  $C_3(D_1, D_2)$ .

of our results appeared in [4] and [8].

The rest of the paper is organized as follows. A summary of prior results pertaining to the Gray-Wyner network and the two notions of CI is given in Section II. We define a lossy generalization of Wyner's CI and derive a single letter information theoretic characterization in Section III. Next, we specialize to the lossy CI of two correlated Gaussian random variables using the information theoretic characterization. In Section IV, we extend the Gács and Körner CI definition to the lossy framework. In Section V, we study the tradeoff between the sum rate and receive rate in the Gray-Wyner network and show that a corresponding contour of points on the boundary of the Gray-Wyner region emerges, which passes through both the CI operating points of Wyner and Gács-Körner.

## II. PRIOR RESULTS

### A. The Gray-Wyner Network [3]

Let  $(X, Y)$  be any two dependent random variables taking values in the alphabets  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively. Let  $\hat{\mathcal{X}}$  and  $\hat{\mathcal{Y}}$  be the respective reconstruction alphabets. For any positive integer  $M$ , we denote the set  $\{1, 2, \dots, M\}$  by  $I_M$ . A sequence of  $n$  independent and identically distributed (iid) random variables is denoted by  $X^n$  and the corresponding alphabet by  $\mathcal{X}^n$ . In what follows, for any pair of random variables  $X$  and  $Y$ ,  $R_X(\cdot)$ ,  $R_Y(\cdot)$  and  $R_{X,Y}(\cdot, \cdot)$  denote the respective rate distortion functions. With slight abuse of notation, we use  $H(\cdot)$  to denote the entropy of a discrete random variable or the differential entropy of a continuous random variable.

A rate-distortion tuple  $(R_0, R_1, R_2, D_1, D_2)$  is said to be achievable by the Gray-Wyner network if for all  $\epsilon > 0$ , there exists encoder and decoder mappings:

$$\begin{aligned} f_E &: \mathcal{X}^n \times \mathcal{Y}^n \rightarrow I_{M_0} \times I_{M_1} \times I_{M_2} \\ f_D^{(X)} &: I_{M_0} \times I_{M_1} \rightarrow \hat{\mathcal{X}}^n \\ f_D^{(Y)} &: I_{M_0} \times I_{M_2} \rightarrow \hat{\mathcal{Y}}^n \end{aligned} \quad (2)$$

where  $f_E(X^n, Y^n) = (S_0, S_1, S_2)$ ,  $S_0 \in I_{M_0}$ ,  $S_1 \in I_{M_1}$  and  $S_2 \in I_{M_2}$ , such that the following hold:

$$\begin{aligned} M_i &\leq 2^{n(R_i + \epsilon)}, \quad i \in \{0, 1, 2\} \\ \Delta_X &\leq D_1 + \epsilon \\ \Delta_Y &\leq D_2 + \epsilon \end{aligned} \quad (3)$$

where  $\hat{X}^n = f_D^{(X)}(S_0, S_1)$ ,  $\hat{Y}^n = f_D^{(Y)}(S_0, S_2)$  and

$$\begin{aligned} \Delta_X &= \frac{1}{n} \sum_{i=1}^n d_X(X_i, \hat{X}_i) \\ \Delta_Y &= \frac{1}{n} \sum_{i=1}^n d_Y(Y_i, \hat{Y}_i) \end{aligned} \quad (4)$$

for some well defined single letter distortion measures  $d_X(\cdot, \cdot)$  and  $d_Y(\cdot, \cdot)$ . The convex closure over all such achievable rate-distortion tuples is called the achievable region for the Gray-Wyner network. The set of all achievable rate tuples for any given distortion  $D_1$  and  $D_2$  is denoted here by  $\mathcal{R}_{GW}(D_1, D_2)$ . We denote the lossless Gray-Wyner region,

as defined in [3], simply by  $\mathcal{R}_{GW}$ , when  $X$  and  $Y$  are random variables with finite entropy.

Gray and Wyner [3] gave the following complete characterization for  $\mathcal{R}_{GW}(D_1, D_2)$ . Let  $(U, \hat{X}, \hat{Y})$  be any random variables jointly distributed with  $(X, Y)$  and taking values in alphabets  $\mathcal{U}, \hat{\mathcal{X}}$  and  $\hat{\mathcal{Y}}$ , respectively (for any arbitrary  $\mathcal{U}$ ). Let the joint density be  $P(X, Y, U, \hat{X}, \hat{Y})$ . All rate-distortion tuples  $(R_0, R_1, R_2, D_1, D_2)$  satisfying the following conditions are achievable:

$$\begin{aligned} R_0 &\geq I(X, Y; U) \\ R_1 &\geq I(X; \hat{X}|U) \\ R_2 &\geq I(Y; \hat{Y}|U) \\ D_1 &\geq E(d_X(X, \hat{X})) \\ D_2 &\geq E(d_Y(Y, \hat{Y})) \end{aligned} \quad (5)$$

The closure of the achievable rate distortion tuples over all such joint densities is the complete rate-distortion region for the Gray-Wyner network,  $\mathcal{R}_{GW}(D_1, D_2)$ . For the lossless framework, the above characterization simplifies significantly. Let  $U$  be any random variable jointly distributed with  $(X, Y)$ . Then, all rate tuples satisfying the following conditions belong to the lossless Gray-Wyner region:

$$\begin{aligned} R_0 &\geq I(X, Y; U) \\ R_1 &\geq H(X|U) \\ R_2 &\geq H(Y|U) \end{aligned} \quad (6)$$

The convex closure of achievable rates, over all such joint densities is denoted by  $\mathcal{R}_{GW}$ .

### B. Wyner's Common Information

Wyner's CI, denoted by  $C_W(X, Y)$ , is defined as:

$$C_W(X, Y) = \inf I(X, Y; U) \quad (7)$$

where the infimum is over all random variables  $U$  such that  $X \leftrightarrow U \leftrightarrow Y$  form a Markov chain in that order. Wyner showed that  $C_W(X, Y)$  is equal to the minimum rate on the shared branch of the GW network, while the total sum rate is constrained to be the joint entropy. To formally state the result, we first define the set  $\mathcal{R}_W$ . A common rate  $R_0$  is said to belong to  $\mathcal{R}_W$  if for any  $\epsilon > 0$ , there exists a point  $(R_0, R_1, R_2)$  such that:

$$\begin{aligned} (R_0, R_1, R_2) &\in \mathcal{R}_{GW} \\ R_0 + R_1 + R_2 &\leq H(X, Y) + \epsilon \end{aligned} \quad (8)$$

Then, Wyner showed that:

$$C_W(X, Y) = \inf R_0 \in \mathcal{R}_W \quad (9)$$

It is worthwhile noting that, if the random variables  $(X, Y)$  are such that every point on the plane  $R_0 + R_1 + R_2 = H(X, Y)$  satisfies (6) with equality for some joint density  $P(X, Y, U)$ , then (8) can be simplified by setting  $\epsilon = 0$ , without loss in optimality. We again note that Wyner showed that the quantity  $C_W(X, Y)$  has other operational interpretations, besides the Gray-Wyner network. Their relations to the lossy generalization we define in this paper are less obvious and will be considered as part of our future work.

### C. The Gács and Körner Common Information

Let  $X$  and  $Y$  be two dependent random variables taking values on finite alphabets  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively. Let  $X^n$  and  $Y^n$  be  $n$  independent copies of  $X$  and  $Y$ . Gács and Körner defined CI of two random variables as follows:

$$C_{GK}(X, Y) = \sup \frac{1}{n} H(f_1(X^n)) \quad (10)$$

where sup is taken over all sequences of functions  $f_1^{(n)}, f_2^{(n)}$ , such that  $P(f_1(X^n) \neq f_2(Y^n)) \rightarrow 0$ . It can be understood as the maximum rate of the codeword that can be generated individually at two encoders observing  $X^n$  and  $Y^n$  separately. To describe their main result, we need the following definition.

*Definition 1:* Without loss of generality, we assume  $P(X = x) > 0 \forall x \in \mathcal{X}$  and  $P(Y = y) > 0, \forall y \in \mathcal{Y}$ . Ergodic decomposition of the stochastic matrix of conditional probabilities  $P(X = x|Y = y)$ , is defined by a partition of the space  $\mathcal{X} \times \mathcal{Y}$  into disjoint subsets,  $\mathcal{X} \times \mathcal{Y} = \bigcup_j \mathcal{X}_j \times \mathcal{Y}_j$ , such that  $\forall j$ :

$$\begin{aligned} P(X = x|Y = y) &= 0 \quad \forall x \in \mathcal{X}_j, \quad y \notin \mathcal{Y}_j \\ P(Y = y|X = x) &= 0 \quad \forall y \in \mathcal{Y}_j, \quad x \notin \mathcal{X}_j \end{aligned} \quad (11)$$

Observe that, the ergodic decomposition will always be such that if  $x \in \mathcal{X}_j$ , then  $Y$  must take values in  $\mathcal{Y}_j$  and vice-versa, i.e., if  $y \in \mathcal{Y}_j$ , then  $X$  must take values only in  $\mathcal{X}_j$ . Let us define the random variable  $J$  as:

$$J = j \text{ iff } x \in \mathcal{X}_j \Leftrightarrow y \in \mathcal{Y}_j \quad (12)$$

Gács and Körner showed that  $C_{GK}(X, Y) = H(J)$ .

The original definition of CI, due to Gács and Körner, was naturally unrelated to the Gray-Wyner network, which it predates. However, an alternate and insightful characterization of  $C_{GK}(X, Y)$  in terms of  $\mathcal{R}_{GW}$ , was given by Ahlswede and Körner in [9] (and also recently appeared in [5]). To formally state the result, we define the set  $\mathcal{R}_{GK}$ . A common rate  $R_0$  is said to belong to  $\mathcal{R}_{GK}$  if for any  $\epsilon > 0$ , there exists a point  $(R_0, R_1, R_2)$  such that:

$$(R_0, R_1, R_2) \in \mathcal{R}_{GW} \quad (13)$$

$$\begin{aligned} R_0 + R_1 &\leq H(X) + \epsilon \\ R_0 + R_2 &\leq H(Y) + \epsilon \end{aligned} \quad (14)$$

Then:

$$C_{GK}(X, Y) = \sup R_0 \in \mathcal{R}_{GK}$$

Specifically, Ahlswede and Körner showed that:

$$H(J) = C_{GK}(X, Y) = \sup I(X, Y; U) \quad (15)$$

subject to,

$$Y \leftrightarrow X \leftrightarrow U \text{ and } X \leftrightarrow Y \leftrightarrow U \quad (16)$$

This characterization of  $C_{GK}(X, Y)$  in terms of the Gray-Wyner network offers a crisp understanding of the difference in objective between the two approaches. It will become evident in Section V that the two are in fact instances of a more general objective, within a broader unified framework. Again it is worthwhile noting that, if every point on the intersection of

the planes  $R_0 + R_1 = H(X)$  and  $R_0 + R_2 = H(Y)$  satisfies (6) with equality for some joint density  $P(X, Y, U)$ , then the above definition can be simplified by setting  $\epsilon = 0$ .

### III. LOSSY EXTENSION OF WYNER'S COMMON INFORMATION

#### A. Definition

We generalize Wyner's CI to the lossy framework and denote it by  $C_W(X, Y; D_1, D_2)$ . Let  $\mathcal{R}_W(D_1, D_2)$  be the set of all  $R_0$  such that, for any  $\epsilon > 0$ , there exists a point  $(R_0, R_1, R_2)$  satisfying the following conditions:

$$\begin{aligned} (R_0, R_1, R_2) &\in \mathcal{R}_{GW}(D_1, D_2) \\ R_0 + R_1 + R_2 &\leq R_{X,Y}(D_1, D_2) + \epsilon \end{aligned}$$

Then, the lossy generalization of Wyner's CI is defined as the infimum over all such shared rates  $R_0$ , i.e.,

$$C_W(X, Y; D_1, D_2) = \inf R_0 \in \mathcal{R}_W(D_1, D_2)$$

Note that, for any distortion pair  $(D_1, D_2)$ , if every point on the plane  $R_0 + R_1 + R_2 = R_{X,Y}(D_1, D_2)$  satisfies (5) with equality for some joint density  $P(X, Y, U, \hat{X}, \hat{Y})$ , then the above definition can be simplified by setting  $\epsilon = 0$ . We note that the plane  $R_0 + R_1 + R_2 = R_{X,Y}(D_1, D_2)$  is called the Pangloss plane in the literature [3]. We note that the above operational definition of  $C_W(X, Y; D_1, D_2)$ , has also appeared recently in [7], albeit without a single letter information theoretic characterization. The primary objective of Section III-B is to characterize  $C_W(X, Y; D_1, D_2)$  for general sources and distortion measures. Wyner gave the complete single letter characterization of  $C_W(X, Y; 0, 0)$  when  $X$  and  $Y$  have finite joint entropy and the distortion measure is Hamming distortion ( $d(x, \hat{x}) = \mathbf{1}_{x \neq \hat{x}}$ ,  $d(y, \hat{y}) = \mathbf{1}_{y \neq \hat{y}}$ ), i.e.,

$$C_W(X, Y) = C_W(X, Y; 0, 0) = \inf I(X, Y; U) \quad (17)$$

where the infimum is over all  $U$  satisfying  $X \leftrightarrow U \leftrightarrow Y$ .

#### B. Single Letter Characterization of $C_W(X, Y; D_1, D_2)$

To simplify the exposition and to make the proof more intuitive, in the following theorem, we assume that for any distortion pair  $(D_1, D_2)$ , every point on the Pangloss plane (i.e.,  $R_0 + R_1 + R_2 = R_{X,Y}(D_1, D_2)$ ) satisfies (5) with equality for some joint density  $P(X, Y, U, \hat{X}, \hat{Y})$ . We handle the more general setting in Appendix A.

*Theorem 1: A single letter characterization of  $C_W(X, Y; D_1, D_2)$  is given by:*

$$C_W(X, Y; D_1, D_2) = \inf I(X, Y; U) \quad (18)$$

where the infimum is over all joint densities  $P(X, Y, \hat{X}, \hat{Y}, U)$  such that the following Markov conditions hold:

$$\begin{aligned} \hat{X} &\leftrightarrow U \leftrightarrow \hat{Y} \\ (X, Y) &\leftrightarrow (\hat{X}, \hat{Y}) \leftrightarrow U \end{aligned} \quad (19)$$

and where  $P(\hat{X}, \hat{Y}|X, Y) \in \mathcal{P}_{D_1, D_2}^{X, Y}$  is any joint distribution which achieves the rate distortion function at  $(D_1, D_2)$ , i.e.,  $I(X, Y; \hat{X}, \hat{Y}) = R_{X,Y}(D_1, D_2)$ ,  $E(d_X(X, \hat{X})) \leq D_1$  and  $E(d_Y(Y, \hat{Y})) \leq D_2$ ,  $\forall P(\hat{X}, \hat{Y}|X, Y) \in \mathcal{P}_{D_1, D_2}^{X, Y}$ .

*Remark 1:* If we set  $\hat{X} = X$ ,  $\hat{Y} = Y$  and consider the Hamming distortion measure, at  $(D_1, D_2) = (0, 0)$ , it is easy to show that Wyner's CI is obtained as a special case, i.e.,  $C_W(X, Y; 0, 0) = C_W(X, Y)$ .

*Proof:* We note that, although there are arguably simpler methods to prove this theorem, we choose the following approach as it uses only the Gray-Wyner theorem without recourse to any supplementary results. Further, we assume that there exists a unique encoder  $P^*(\hat{X}^*, \hat{Y}^*|X, Y) \in \mathcal{P}_{D_1, D_2}^{X, Y}$  which achieves  $R_{X,Y}(D_1, D_2)$ . The proof of the theorem when there are multiple encoders in  $\mathcal{P}_{D_1, D_2}^{X, Y}$  follows directly.

Our objective is to show that every point in the intersection of  $\mathcal{R}_{GW}(D_1, D_2)$  and the Pangloss plane has  $R_0 = I(X, Y; U)$  for some  $U$  jointly distributed with  $(X, Y, \hat{X}^*, \hat{Y}^*)$  and satisfying conditions (19) and (20). We first prove that every point in the intersection of the Pangloss plane and  $\mathcal{R}_{GW}(D_1, D_2)$  is achieved by a joint density satisfying (19) and (20). Towards showing this, we begin with an alternate characterization of  $\mathcal{R}_{GW}(D_1, D_2)$  (which is also complete) due to Venkataramani et al. (see section III.B in [10]).<sup>2</sup> Let  $(U, \hat{X}, \hat{Y})$  be any random variables jointly distributed with  $(X, Y)$  such that  $E(d_X(X, \hat{X})) \leq D_1$  and  $E(d_Y(Y, \hat{Y})) \leq D_2$ . Then any rate tuple  $(R_0, R_1, R_2)$  satisfying the following conditions belongs to  $\mathcal{R}_{GW}(D_1, D_2)$ :

$$\begin{aligned} R_0 &\geq I(X, Y; U) \\ R_1 + R_0 &\geq I(X, Y; U, \hat{X}) \\ R_2 + R_0 &\geq I(X, Y; U, \hat{Y}) \\ R_0 + R_1 + R_2 &\geq I(X, Y; U, \hat{X}, \hat{Y}) + I(\hat{X}; \hat{Y}|U) \end{aligned} \quad (21)$$

It is easy to show that the above characterization is equivalent to (5). As the above characterization is complete, this implies that, if a rate-distortion tuple  $(R_0, R_1, R_2, D_1, D_2)$  is achievable for the Gray-Wyner network, then we can always find random variables  $(U, \hat{X}, \hat{Y})$  such that  $E(d_X(X, \hat{X})) \leq D_1$ ,  $E(d_Y(Y, \hat{Y})) \leq D_2$  and satisfying (21). We are further interested in characterizing the points in  $\mathcal{R}_{GW}(D_1, D_2)$  that lie on the Pangloss plane, i.e.,  $R_0 + R_1 + R_2 = R_{X,Y}(D_1, D_2)$ . Therefore, for any rate tuple  $(R_0, R_1, R_2)$  on the Pangloss plane in  $\mathcal{R}_{GW}(D_1, D_2)$ , we have the following series of inequalities:

$$\begin{aligned} R_{X,Y}(D_1, D_2) &= R_0 + R_1 + R_2 \\ &\geq I(X, Y; U, \hat{X}, \hat{Y}) + I(\hat{X}; \hat{Y}|U) \\ &\geq I(X, Y; \hat{X}, \hat{Y}) + I(\hat{X}; \hat{Y}|U) \\ &\stackrel{(a)}{\geq} R_{X,Y}(D_1, D_2) + I(\hat{X}; \hat{Y}|U) \\ &\geq R_{X,Y}(D_1, D_2) \end{aligned} \quad (22)$$

where (a) follows because  $(\hat{X}, \hat{Y})$  satisfy the distortion constraints. Since the above chain of inequalities start and end with the same quantity, they must all be identities and we

<sup>2</sup>We note that the theorem can be proved even using the original Gray-Wyner characterization. However, if we begin with that characterization, we would require the random variables to satisfy two additional Markov conditions beyond (19) and (20). These Markov conditions can in fact be shown to be redundant from the Kuhn-Tucker conditions. The alternate approach we choose circumvents these supplementary arguments.

have:

$$\begin{aligned} I(\hat{X}; \hat{Y}|U) &= 0 \\ I(X, Y; U, \hat{X}, \hat{Y}) &= I(X, Y; \hat{X}, \hat{Y}) \\ I(X, Y; \hat{X}, \hat{Y}) &= R_{X,Y}(D_1, D_2) \end{aligned} \quad (23)$$

By assumption, there is a unique encoder,  $P^*(\hat{X}^*, \hat{Y}^*|X, Y)$ , which achieves  $I(X, Y; \hat{X}, \hat{Y}) = R_{X,Y}(D_1, D_2)$ . It therefore follows that every point in  $\mathcal{R}_{GW}(D_1, D_2)$  that lies on the Pangloss plane satisfies (21) for some joint density satisfying (19) and (20).

It remains to be shown that any joint density  $(X, Y, \hat{X}^*, \hat{Y}^*, U)$  satisfying (19) and (20) leads to a sub-region of  $\mathcal{R}_{GW}(D_1, D_2)$  which has at least one point on the Pangloss plane with  $R_0 = I(X, Y; U)$ . Formally, denote by  $\mathcal{R}(U)$ , the region (21) achieved by a joint density  $(X, Y, \hat{X}^*, \hat{Y}^*, U)$  satisfying (19) and (20). We need to show that  $\exists(R_0, R_1, R_2) \in \mathcal{R}(U)$  such that:

$$\begin{aligned} R_0 + R_1 + R_2 &= R_{X,Y}(D_1, D_2) \\ R_0 &= I(X, Y; U) \end{aligned} \quad (24)$$

Consider the point,  $(R_0, R_1, R_2) = (I(X, Y; U), I(X, Y; \hat{X}^*|U), I(X, Y; \hat{Y}^*|U, \hat{X}^*))$  for any joint density  $(X, Y, \hat{X}^*, \hat{Y}^*, U)$  satisfying (19) and (20). Clearly the point satisfies the first two conditions in (21). Next, we note that:

$$\begin{aligned} R_0 + R_2 &= I(X, Y; U) + I(X, Y; \hat{Y}^*|U, \hat{X}^*) \\ &\stackrel{(b)}{\geq} I(X, Y; U) + I(X, Y; \hat{Y}^*|U) \\ &= I(X, Y; \hat{Y}^*, U) \end{aligned} \quad (25)$$

$$\begin{aligned} R_0 + R_1 + R_2 &\stackrel{(c)}{=} I(X, Y; \hat{X}^*, \hat{Y}^*, U) \\ &= I(X, Y; \hat{X}^*, \hat{Y}^*) = R_{X,Y}(D_1, D_2) \end{aligned} \quad (26)$$

where (b) and (c) follow from the fact that the joint density satisfies (19) and (20). Hence, we have shown the existence of one point in  $\mathcal{R}(U)$  satisfying (24) for every joint density  $(X, Y, \hat{X}^*, \hat{Y}^*, U)$  satisfying (19) and (20), proving the theorem. ■

The following corollary sheds light on several properties related to the optimizing random variables  $U$  in Theorem 1. These properties significantly simplify the computation of lossy CI.

*Corollary 1: The joint distribution that optimizes (18) in Theorem 1 satisfies the following properties:*

$$\begin{aligned} \hat{X}^* &\leftrightarrow (X, Y, U) \leftrightarrow \hat{Y}^* \\ \hat{X}^* &\leftrightarrow (X, U) \leftrightarrow Y \\ \hat{Y}^* &\leftrightarrow (Y, U) \leftrightarrow X \end{aligned} \quad (27)$$

*i.e., the conditional density can always be written as:*

$$P(\hat{X}^*, \hat{Y}^*, U|X, Y) = P(\hat{X}^*, U|X)P(\hat{Y}^*, U|Y) \quad (28)$$

*Proof:* We relegate the proof to Appendix B as it is quite orthogonal to the main flow of the paper. ■

We note that, in general,  $C_W(X, Y; D_1, D_2)$  is neither convex/concave nor monotonic with respect to  $(D_1, D_2)$ . As we will see later,  $C_W(X, Y; D_1, D_2)$  is non-monotonic even for two correlated Gaussian random variables under mean squared distortion measure. This makes it hard to establish

conclusive inequality relations between  $C_W(X, Y; D_1, D_2)$  and  $C_W(X, Y)$  for all distortions. However, in the following lemma, we establish sufficient conditions on  $(D_1, D_2)$  for  $C_W(X, Y; D_1, D_2) \leq C_W(X, Y)$ . In Appendix C, we review some of the results pertinent to Shannon lower bounds for vectors of random variables, that will be useful in the following Lemma.

*Lemma 1: For any pair of random variables  $(X, Y)$ ,*

- (i)  $C_W(X, Y; D_1, D_2) \leq C_W(X, Y)$  at  $(D_1, D_2)$  if  $\exists(\bar{D}_1, \bar{D}_2)$  such that  $\bar{D}_1 \leq D_1$ ,  $\bar{D}_2 \leq D_2$  and  $R_{X,Y}(\bar{D}_1, \bar{D}_2) = C_W(X, Y)$ .
- (ii) For any difference distortion measures,  $C_W(X, Y; D_1, D_2) \geq C_W(X, Y)$ , if the Shannon lower bound for  $R_{XY}(D_1, D_2)$  is tight at  $(D_1, D_2)$ .

*Proof:* The proof of (i) is straightforward and hence omitted. Towards proving (ii), we appeal to standard techniques [11], [12] (also refer to Appendix C) which immediately show that the conditional distribution  $P(\hat{X}^*, \hat{Y}^*|X, Y)$  that achieves  $R_{X,Y}(D_1, D_2)$  when Shannon lower bound is tight, has independent backward channels, i.e.:

$$P_{X,Y|\hat{X}^*, \hat{Y}^*}(x, y|\hat{x}^*, \hat{y}^*) = Q_{X|\hat{X}^*}(x|\hat{x}^*)Q_{Y|\hat{Y}^*}(y|\hat{y}^*)$$

Let us consider any  $U$  that satisfies  $(\hat{X}^* \leftrightarrow U \leftrightarrow \hat{Y}^*)$  and  $(X, Y) \leftrightarrow (\hat{X}^*, \hat{Y}^*) \leftrightarrow U$ . It is easy to verify that any such joint density also satisfies  $X \leftrightarrow U \leftrightarrow Y$ . As the infimum for  $C_W(X, Y)$  is taken over a larger set of joint densities, we have  $C_W(X, Y; D_1, D_2) \geq C_W(X, Y)$ . ■

The above lemma highlights the anomalous behavior of  $C_W(X, Y; D_1, D_2)$  with respect to the distortions. Determining the conditions for equality in Lemma 1. (ii) is an interesting problem in its own right. It was shown in [7] that there always exists a region of distortions around the origin such that  $C_W(X, Y; D_1, D_2) = C_W(X, Y)$ . We will further explore the underlying connections between these results as part of future work.

### C. Bivariate Gaussian Example

Let  $X$  and  $Y$  be jointly Gaussian random variables with zero mean, unit variance and a correlation coefficient of  $\rho$ . Let the distortion measure be the mean squared error (MSE), i.e.,  $D_1(X, \hat{X}) = (X - \hat{X})^2$  and  $D_2(Y, \hat{Y}) = (Y - \hat{Y})^2$ .

Hereafter, for simplicity, we assume that  $\rho \in [0, 1]$ , noting that all results can be easily extended to negative values of  $\rho$  with appropriate modifications. The joint rate distortion function is given by (29) at the top of the next page (see [11]), where  $\bar{D}_i = 1 - D_i$ .

Let us first consider the range of distortions such that  $\bar{D}_1 \bar{D}_2 \geq \rho^2$ . The RD-optimal random encoder is such that  $P(X|\hat{X}^*)$  and  $P(Y|\hat{Y}^*)$  are two independent zero mean Gaussian channels with variances  $D_1$  and  $D_2$ , respectively. It is easy to verify that the optimal reproduction distribution (for  $(\hat{X}^*, \hat{Y}^*)$ ) is jointly Gaussian with zero mean. The covariance matrix for  $(\hat{X}^*, \hat{Y}^*)$  is:

$$\Sigma_{\hat{X}^* \hat{Y}^*} = \begin{bmatrix} \bar{D}_1 & \rho \\ \rho & \bar{D}_2 \end{bmatrix} \quad (31)$$

$$R_{X,Y}(D_1, D_2) = \begin{cases} \frac{1}{2} \log \left( \frac{1-\rho^2}{D_1 D_2} \right) & \text{if } \bar{D}_1 \bar{D}_2 \geq \rho^2 \\ \frac{1}{2} \log \left( \frac{1-\rho^2}{D_1 D_2 - (\rho - \sqrt{D_1 D_2})^2} \right) & \text{if } \bar{D}_1 \bar{D}_2 \leq \rho^2, \min \left\{ \frac{\bar{D}_1}{D_2}, \frac{\bar{D}_2}{D_1} \right\} \geq \rho^2 \\ \frac{1}{2} \log \left( \frac{1}{\min(D_1, D_2)} \right) & \text{if } \bar{D}_1 \bar{D}_2 \leq \rho^2, \min \left\{ \frac{\bar{D}_1}{D_2}, \frac{\bar{D}_2}{D_1} \right\} < \rho^2 \end{cases} \quad (29)$$

$$C_W(X, Y, D_1, D_2) = \begin{cases} C_W(X, Y) = \frac{1}{2} \log \frac{1+\rho}{1-\rho} & \text{if } \max\{D_1, D_2\} \leq 1-\rho \\ R_{X,Y}(D_1, D_2) & \text{if } \bar{D}_1 \bar{D}_2 \leq \rho^2 \\ \frac{1}{2} \log \frac{1-\rho^2}{(1-\frac{\rho^2}{D_1}) D_1} & \text{if } D_1 > 1-\rho, \bar{D}_1 \bar{D}_2 > \rho^2 \\ \frac{1}{2} \log \frac{1-\rho^2}{(1-\frac{\rho^2}{D_2}) D_2} & \text{if } D_2 > 1-\rho, \bar{D}_1 \bar{D}_2 > \rho^2 \end{cases} \quad (30)$$

Observe that, at these distortions, the Shannon lower bound is tight. Next, let us consider the range of distortions such that  $\bar{D}_1 \bar{D}_2 \leq \rho^2$ . The RD-optimal random encoder in this distortion range is such that  $\hat{X}^* = a\hat{Y}^*$  and the conditional distribution  $P(X, Y | \hat{X}^* = a\hat{Y}^*)$  is jointly Gaussian with correlated components, where:

$$a = \begin{cases} \sqrt{\frac{\bar{D}_1}{D_2}} & \min \left\{ \frac{\bar{D}_1}{D_2}, \frac{\bar{D}_2}{D_1} \right\} \geq \rho^2 \\ \rho & \min \left\{ \frac{\bar{D}_1}{D_2}, \frac{\bar{D}_2}{D_1} \right\} < \rho^2 \end{cases} \quad (32)$$

Clearly, the Shannon lower bound is not tight in this regime of distortions. Note that the regime of distortions where  $\bar{D}_1 \bar{D}_2 \leq \rho^2$  and  $\min \left\{ \frac{\bar{D}_1}{D_2}, \frac{\bar{D}_2}{D_1} \right\} < \rho^2$ , is degenerate in the sense that one of the two distortions can be reduced without incurring any excess sum-rate [13].

It was shown in [7] that, for two correlated Gaussian random variables,  $C_W(X, Y) = \frac{1}{2} \log \frac{1+\rho}{1-\rho}$  and the infimum achieving  $U^*$  is a standard Gaussian random variable jointly distributed with  $(X, Y)$  as:

$$\begin{aligned} X &= \sqrt{\rho}U^* + \sqrt{1-\rho}N_1 \\ Y &= \sqrt{\rho}U^* + \sqrt{1-\rho}N_2 \end{aligned} \quad (33)$$

where  $N_1, N_2$  and  $U^*$  are independent standard Gaussian random variables. Equipped with these results, we derive in the following theorem, the lossy CI of two correlated Gaussian random variables, and then demonstrate its anomalous behavior.

*Theorem 2: The lossy CI of two correlated zero-mean Gaussian random variables with unit variance and correlation coefficient  $\rho$ , is given by (30) at the top of the page.*

*Remark 2:* The different regimes of distortions indicated in (30) are depicted in Fig. 2(a). The lossy CI is equal to the corresponding lossless information theoretic characterization in the regime where both  $D_1$  and  $D_2$  are smaller than  $1-\rho$ . In the regime where  $\bar{D}_1 \bar{D}_2 \leq \rho^2$ , the lossy CI is equal to the joint rate-distortion function, i.e., all the bits are sent on the shared branch. In the other two regimes, the lossy CI is strictly greater than the lossless characterization, i.e.,  $C_W(X, Y; D_1, D_2) > C_W(X, Y)$ . To illustrate, we fix  $\rho = 0.5$  and  $D_2 = 0.2$ , and plot the  $C_W(X, Y; 0.2, D_2)$  as a function of  $D_1$ , as shown in Fig. 2(b). Observe that the lossy CI remains a constant till point  $A$ , where  $D_1 = 1-\rho$ . It is strictly greater than

the lossless characterization between points  $A$  and  $B$ , i.e., between  $D_1 = 1-\rho$  and  $D_1 = 1-\frac{\rho}{1-D_1}$  and finally is equal to the joint rate-distortion function for all points to the right of  $B$ . This result is quite counter-intuitive and reveals a surprising property of the Gray-Wyner network. Note that while traversing from the origin to point  $A$ , the lossy CI is a constant, and decreasing only the rates of the side branches is optimal in achieving the minimum sum rate at the respective distortions. However, between points  $A$  and  $B$ , the rate on the common branch *increases*, while the rates on the side branches continue to decrease in order to maintain sum rate optimality at the respective distortions. This implies that, even a Gaussian source under mean squared error distortion measure, one of the simplest successively refinable examples in point to point settings, is not successively refinable on the Gray-Wyner network, i.e, the bit streams on the three branches, when  $D_1$  is set to a value in between  $(A, B)$ , are not subsets of the respective bit-streams required to achieve a distortion in between the origin and point  $A$ . Finally note that, for all points to the right of  $B$ , all the information is carried on the common branch and the side branches are left unused, i.e., to achieve minimum sum rate, all the bits must be transmitted on the shared branch. This example clearly demonstrates that the lossy CI is neither convex/concave nor is monotonic in general.

Before stating the formal proof, we provide a high level intuitive argument to justify the non-monotone behavior of  $C_W(X, Y; D_1, D_2)$ . First, it follows from results in [7] that there exists a region of distortions around origin such that  $C_W(X, Y; D_1, D_2) = C_W(X, Y)$ , where the optimizing  $U$  in Theorem 1 is equal to  $U^*$ . Next, recall that, if  $\bar{D}_1 \bar{D}_2 \geq \rho^2$ , Shannon lower bound is tight and hence from Lemma 1,  $C_W(X, Y; D_1, D_2) \geq C_W(X, Y)$ . For  $C_W(X, Y; D_1, D_2)$  to be equal to  $C_W(X, Y)$ , there should exist a joint density over  $(X, Y, \hat{X}^*, \hat{Y}^*, U^*)$ , such that,  $(X, Y, U^*)$  is distributed according to (33) and the Markov chain condition  $X \leftrightarrow \hat{X}^* \leftrightarrow U^* \leftrightarrow \hat{Y}^* \leftrightarrow Y$  is satisfied. However, if  $D_1 > 1-\rho$  and  $\bar{D}_1 \bar{D}_2 \geq \rho^2$ ,  $I(X; \hat{X}^*) = \frac{1}{2} \log \frac{1}{D_1} < \frac{1}{2} \log \frac{1}{1-\rho} = I(X; U^*)$ . Therefore, it is impossible to find such a joint density and hence  $C_W(X, Y; D_1, D_2) > C_W(X, Y)$ . Finally, note that if  $\bar{D}_1 \bar{D}_2 \leq \rho^2$ ,  $\hat{X}^* = a\hat{Y}^*$  and hence  $U^*$  must be equal to  $X^*$  to satisfy  $\hat{X}^* \leftrightarrow U^* \leftrightarrow \hat{Y}^*$ . This implies that  $C_W(X, Y; D_1, D_2) = R_{X,Y}(D_1, D_2)$ , which is a monotonically decreasing function. The above

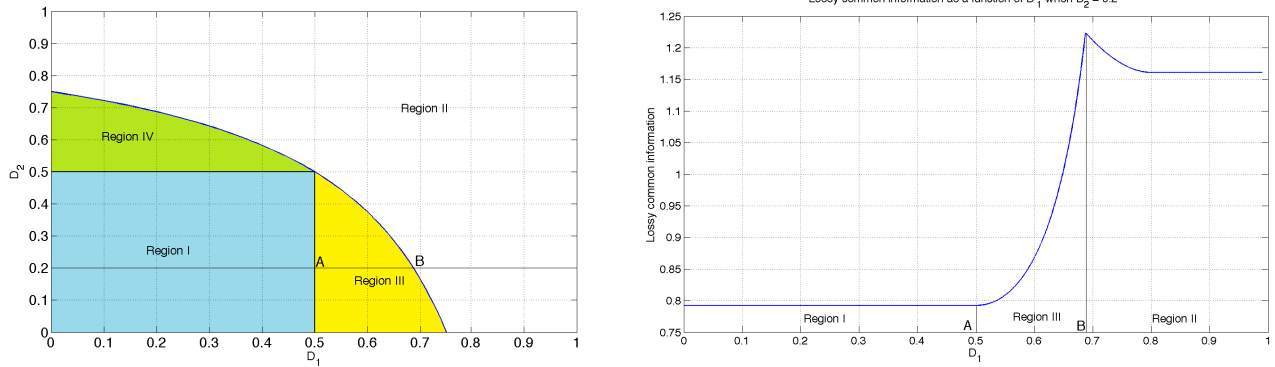


Fig. 2. The figure on the left shows the different regions of distortions indicated in (30) when  $\rho = 0.5$ . Region I is blue, region II is white, region III is yellow and region IV is green. The figure on the right shows the lossy CI for two correlated Gaussian random variables at a fixed  $D_2 = 0.2$  and as a function of  $D_1$ . Observe that the curve is equal to the lossless CI till  $D_1 = 1 - \rho$ , then increases till  $D_1 = 1 - \frac{\rho^2}{1 - D_2}$  and then finally drops off as  $R_{X,Y}(D_1, D_2)$ .

arguments clearly demonstrate the non-monotone behavior of  $C_W(X, Y; D_1, D_2)$ .

*Proof:* We first consider the regime of distortions where  $\max\{D_1, D_2\} \leq 1 - \rho$ . At these distortions, the Shannon lower bound is tight and hence by Lemma 1,  $C_W(X, Y; D_1, D_2) \geq C_W(X, Y)$ . To prove that  $C_W(X, Y; D_1, D_2) = C_W(X, Y)$ , it is sufficient for us to show the existence of a joint distribution over  $(X, Y, \hat{X}^*, \hat{Y}^*, U^*)$  satisfying (19) and (20), where  $(X, Y, U^*)$  satisfy (33) and  $(\hat{X}^*, \hat{Y}^*)$  achieve joint RD optimality at  $(D_1, D_2)$ . We can generate  $(\hat{X}^*, \hat{Y}^*)$  by passing  $U^*$  through independent Gaussian channels as follows:

$$\begin{aligned} \hat{X}^* &= \sqrt{\rho}U^* + \sqrt{1 - D_1 - \rho}\tilde{N}_1 \\ \hat{Y}^* &= \sqrt{\rho}U^* + \sqrt{1 - D_2 - \rho}\tilde{N}_2 \end{aligned} \quad (34)$$

where  $\tilde{N}_1$  and  $\tilde{N}_2$  are independent standard Gaussian random variables independent of both  $N_1$  and  $N_2$ . Therefore there exists a joint density over  $(X, Y, \hat{X}^*, \hat{Y}^*, U^*)$  satisfying  $\hat{X}^* \leftrightarrow U^* \leftrightarrow \hat{Y}^*$  and  $(X, Y) \leftrightarrow (\hat{X}^*, \hat{Y}^*) \leftrightarrow U^*$ . This shows that  $C_W(X, Y; D_1, D_2) \leq C_W(X, Y)$ . Therefore in the range  $\max\{D_1, D_2\} \leq 1 - \rho$ , we have  $C_W(X, Y; D_1, D_2) = C_W(X, Y)$ . We note that, for the symmetric case with  $D_1 = D_2 \leq 1 - \rho$ , this specific result was already deduced in [7], albeit using an alternate approach deriving results from conditional rate-distortion theory.

We next consider the range of distortions where  $\bar{D}_1\bar{D}_2 \leq \rho^2$ . Note that the Shannon lower bound for  $R_{X,Y}(D_1, D_2)$  is not tight in this range. However, the RD-optimal conditional distribution  $P(\hat{X}^*, \hat{Y}^*|X, Y)$  in this distortion range is such that  $\hat{X}^* = a\hat{Y}^*$ , for some constant  $a$ . Therefore the only  $U$  that satisfies  $(\hat{X}^* \leftrightarrow U \leftrightarrow \hat{Y}^*)$  is  $U = \hat{X}^* = a\hat{Y}^*$ . Therefore by Theorem 1, we conclude that  $C_W(X, Y; D_1, D_2) = R_{X,Y}(D_1, D_2)$  for  $\bar{D}_1\bar{D}_2 \leq \rho^2$ . We note that, if either  $D_1$  or  $D_2$  is greater than 1, then no information needs to be sent from the encoder to either one of the two decoders, i.e.:

$$R_{X,Y}(D_1, D_2) = \begin{cases} R_X(D_1) & \text{if } D_1 < 1, D_2 > 1 \\ R_Y(D_2) & \text{if } D_1 > 1, D_2 < 1 \end{cases} \quad (35)$$

In the Gray-Wyner network, these bits can be sent only on the respective private branches, and hence the minimum rate on

the shared branch can be made 0 while achieving a sum rate of  $R_{X,Y}(D_1, D_2)$ . Therefore, if  $D_1$  or  $D_2$  is greater than 1,  $C_W(X, Y; D_1, D_2) = 0$ .

Next, let us consider the third regime of distortions wherein  $\max\{D_1, D_2\} > 1 - \rho$  and  $\bar{D}_1\bar{D}_2 > \rho^2$ . This corresponds to the two intermediate regions in Fig. 2(a). It is important to note that the Shannon lower bound is actually tight in this regime. Therefore, we have  $C_W(X, Y; D_1, D_2) \geq C_W(X, Y)$ . First, we show that  $C_W(X, Y; D_1, D_2) > C_W(X, Y)$ , i.e., it is impossible to find a joint density over  $(X, Y, \hat{X}^*, \hat{Y}^*, U^*)$  satisfying (19) and (20), where  $(X, Y, U^*)$  satisfy (33) and  $(\hat{X}^*, \hat{Y}^*)$  achieve joint RD optimality at  $(D_1, D_2)$ . Towards proving this, note that if the joint density over  $(X, Y, U^*)$  satisfies (33), then  $I(X; U^*) = I(Y; U^*) = \frac{1}{2} \log \frac{1}{1 - \rho}$ . Also note that any joint density that satisfies (19) and (20) also satisfies  $X \leftrightarrow \hat{X}^* \leftrightarrow U^*$  and  $Y \leftrightarrow \hat{Y}^* \leftrightarrow U^*$ , and therefore  $I(X; \hat{X}^*) \geq I(X; U^*)$  and  $I(Y; \hat{Y}^*) \geq I(Y; U^*)$ . However, if  $\max\{D_1, D_2\} > 1 - \rho$ , then  $\min\{I(X; \hat{X}^*), I(Y; \hat{Y}^*)\} < \frac{1}{2} \log \frac{1}{1 - \rho}$ . Hence, it follows that there exists no such joint density over  $(X, Y, \hat{X}^*, \hat{Y}^*, U^*)$ , proving that  $C_W(X, Y; D_1, D_2) > C_W(X, Y)$  if  $\max\{D_1, D_2\} > 1 - \rho$  and  $(1 - D_1)(1 - D_2) > \rho^2$ .

Finally, we prove that the lossy CI in this regime of distortions is given by (30). Recall that the RD-optimal random encoder is such that  $P(X, Y|\hat{X}^*, \hat{Y}^*) = P(X|\hat{X}^*)P(Y|\hat{Y}^*)$  and the covariance matrix for  $(\hat{X}^*, \hat{Y}^*)$  is given by (31). Our objective is to find the joint density over  $(X, Y, \hat{X}^*, \hat{Y}^*, U)$  satisfying (19) and (20), that minimizes  $I(X, Y; U)$ . Clearly, this joint density additionally satisfies all the following Markov conditions:

$$\begin{aligned} X &\leftrightarrow \hat{X}^* \leftrightarrow U \\ Y &\leftrightarrow \hat{Y}^* \leftrightarrow U \\ X &\leftrightarrow U \leftrightarrow Y \end{aligned} \quad (36)$$

Hereafter, we restrict ourselves to the regime of distortion where  $D_1 > 1 - \rho$  and  $\bar{D}_1\bar{D}_2 > \rho^2$ . This corresponds to region III in Fig. 2(a). Similar arguments hold for regime IV. Let us consider two extreme possibilities for the random variable  $U$ . Observe that both choices  $U = \hat{X}^*$  and  $U = \hat{Y}^*$  satisfy all the required Markov conditions. In fact, it is easy to



verify that evaluating  $I(X, Y; \hat{X}^*)$  leads to the expression for  $C_W(X, Y; D_1, D_2)$  in (30) for regime III (and correspondingly evaluating  $I(X, Y; \hat{Y}^*)$  leads to the lossy CI in regime IV). Hence, we need to prove that, in regime III, the optimum  $U = \hat{X}^*$ .

First, we rewrite the objective function as follows:

$$\begin{aligned} \inf I(X, Y; U) &= \inf H(X, Y) - H(X, Y|U) \\ &= \inf H(X, Y) - H(X|U) - H(Y|U) \end{aligned} \quad (37)$$

Hence, our objective is equivalent to maximizing  $H(X|U) + H(Y|U)$  subject to (19), (20) and (36). We will next prove that  $U = \hat{X}^*$  is the solution to this problem.

Recall an important result called the data-processing inequality for minimum mean squared error (MMSE) estimation [14], [15]. It states that, if  $X \leftrightarrow Y \leftrightarrow Z$  form a Markov chain, then  $\Phi(X|Y) \leq \Phi(X|Z)$ , where  $\Phi(A|B) = E[(A - E[A|B])^2]$  is the MMSE of estimating  $A$  from  $B$ . Hence, it follows that for any joint density  $P(X, Y, \hat{X}^*, \hat{Y}^*, U)$ , satisfying (36), we have:

$$\begin{aligned} \Phi(X|U) &\geq D_1 \\ \Phi(Y|U) &\geq D_2 \end{aligned} \quad (38)$$

Next, we consider a less constrained optimization problem and prove that the solution to this less constrained problem is bounded by (30). It then follows that the solution to (37) is  $U = \hat{X}^*$ . Consider the following problem:

$$\sup \left\{ H(\tilde{X}|\tilde{U}) + H(\tilde{Y}|\tilde{U}) \right\} \quad (39)$$

where the supremum is over all joint densities  $P(\tilde{X}, \tilde{Y}, \tilde{U})$ , subject to the following conditions:

$$\begin{aligned} (\tilde{X}, \tilde{Y}) &\sim (X, Y) \\ \tilde{X} &\leftrightarrow \tilde{U} \leftrightarrow \tilde{Y} \\ \Phi(\tilde{X}|\tilde{U}) &\geq D_1 \\ \Phi(\tilde{Y}|\tilde{U}) &\geq D_2 \end{aligned} \quad (40)$$

Observe that all the conditions involving  $(\hat{X}^*, \hat{Y}^*)$  have been dropped in the above formulation and the constraints for this problem are a subset of those in (37). We will next show that the optimum for the above less constrained problem leads to the expressions in (30).

Before proceeding, we show that for any three random variables satisfying  $\tilde{X} \leftrightarrow \tilde{U} \leftrightarrow \tilde{Y}$ , where  $\tilde{X}$  and  $\tilde{Y}$  are zero mean and of unit variance, we have:

$$\begin{aligned} (1 - \Phi(\tilde{X}|\tilde{U}))(1 - \Phi(\tilde{Y}|\tilde{U})) &\geq (E(\tilde{X}\tilde{Y}))^2 \\ &= \rho^2 \end{aligned} \quad (41)$$

Denote the optimal estimators  $\theta_{\tilde{X}}(\tilde{U}) = E(\tilde{X}|\tilde{U})$  and  $\theta_{\tilde{Y}}(\tilde{U}) = E(\tilde{Y}|\tilde{U})$ . Then, we have:

$$\begin{aligned} \Phi(\tilde{X}|\tilde{U}) &= E \left[ \left( \tilde{X} - \theta_{\tilde{X}}(\tilde{U}) \right)^2 \right] \\ &= E \left[ \tilde{X}^2 \right] - E \left[ \left( \theta_{\tilde{X}}(\tilde{U}) \right)^2 \right] \\ &= 1 - E \left[ \left( \theta_{\tilde{X}}(\tilde{U}) \right)^2 \right] \end{aligned} \quad (42)$$

Therefore, we have:

$$\begin{aligned} &(1 - \Phi(\tilde{X}|\tilde{U}))(1 - \Phi(\tilde{Y}|\tilde{U})) \\ &= E \left[ \left( \theta_{\tilde{X}}(\tilde{U}) \right)^2 \right] E \left[ \left( \theta_{\tilde{Y}}(\tilde{U}) \right)^2 \right] \\ &\geq^{(a)} \left( E \left[ \theta_{\tilde{X}}(\tilde{U})\theta_{\tilde{Y}}(\tilde{U}) \right] \right)^2 \\ &=^{(b)} \left( E \left[ E \left[ \tilde{X}\tilde{Y} | \tilde{U} \right] \right] \right)^2 \\ &= \left( E \left[ \tilde{X}\tilde{Y} \right] \right)^2 = \rho^2 \end{aligned} \quad (43)$$

where (a) follows from Cauchy-Schwarz inequality and (b) follows from the Markov condition  $\tilde{X} \leftrightarrow \tilde{U} \leftrightarrow \tilde{Y}$ .

This allows us to further simplify the formulation in (39). Specifically, we relax the constraints (40) by imposing (41), instead of the Markov condition  $\tilde{X} \leftrightarrow \tilde{U} \leftrightarrow \tilde{Y}$ . Our objective now becomes:

$$\sup \left\{ H(\tilde{X}|\tilde{U}) + H(\tilde{Y}|\tilde{U}) \right\} \quad (44)$$

where the supremum is over all joint densities  $P(\tilde{X}, \tilde{Y}, U)$ , subject to the following conditions:

$$\begin{aligned} (\tilde{X}, \tilde{Y}) &\sim (X, Y) \\ (1 - \Phi(\tilde{X}|\tilde{U}))(1 - \Phi(\tilde{Y}|\tilde{U})) &\geq \rho^2 \\ \Phi(\tilde{X}|\tilde{U}) &\geq D_1 \\ \Phi(\tilde{Y}|\tilde{U}) &\geq D_2 \end{aligned} \quad (45)$$

We next bound  $H(\tilde{X}|\tilde{U}) + H(\tilde{Y}|\tilde{U})$  in terms of the corresponding *MMSE* as:

$$\begin{aligned} H(\tilde{X}|\tilde{U}) + H(\tilde{Y}|\tilde{U}) &\leq \frac{1}{2} \log \left( 2\pi e \Phi(\tilde{X}|\tilde{U}) \right) \\ &\quad + \frac{1}{2} \log \left( 2\pi e \Phi(\tilde{Y}|\tilde{U}) \right) \end{aligned} \quad (46)$$

Using (46) to bound (44) leads to the following objective function:

$$\sup \left\{ \Phi(\tilde{X}|\tilde{U})\Phi(\tilde{Y}|\tilde{U}) \right\} \quad (47)$$

subject to the conditions in (45). It is easy to verify that the maximum for this objective function satisfying (45) is achieved either at  $(\Phi(\tilde{X}|\tilde{U}) = D_1, \Phi(\tilde{Y}|\tilde{U}) = 1 - \frac{\rho^2}{1-D_1})$  or at  $(\Phi(\tilde{X}|\tilde{U}) = 1 - \frac{\rho^2}{1-D_2}, \Phi(\tilde{Y}|\tilde{U}) = D_2)$ , depending on whether  $D_1 > 1 - \rho$  or  $D_2 > 1 - \rho$ . Substituting these values in (46) leads to upper bounds on  $H(\tilde{X}|\tilde{U}) + H(\tilde{Y}|\tilde{U})$  for the two distortion regimes, respectively. These upper bounds are achieved by setting  $U = \hat{X}^*$  or  $U = \hat{Y}^*$ , depending on the distortion regime. The proof of the theorem follows by noting that these choices for  $U$  lead to the lossy CI being equal to (30). Therefore, we have completely characterized  $C_W(X, Y; D_1, D_2)$  for  $(X, Y)$  jointly Gaussian for all distortions  $(D_1, D_2) > 0$ . ■

#### IV. LOSSY EXTENSION OF THE GÁCS-KÖRNER COMMON INFORMATION

##### A. Definition

Recall the definition of the Gács-Körner CI from Section II. Although the original definition does not have a direct lossy



interpretation, the equivalent definition given by Ahlswede and Körner, in terms of the lossless Gray-Wyner region can be extended to the lossy setting, similar to our approach to Wyner's CI. These generalizations provide theoretical insight into the performance limits of practical databases for fusion storage of correlated sources as described in [4].

We define the lossy generalization of the Gács-Körner CI at  $(D_1, D_2)$ , denoted by  $C_{GK}(X, Y; D_1, D_2)$  as follows. Let  $\mathcal{R}_{GK}(D_1, D_2)$  be the set of  $R_0$  such that for any  $\epsilon > 0$ , there exists a point  $(R_0, R_1, R_2)$  satisfying the following conditions:

$$(R_0, R_1, R_2) \in \mathcal{R}_{GW}(D_1, D_2) \quad (48)$$

$$R_0 + R_1 \leq R_X(D_1) + \epsilon \quad R_0 + R_2 \leq R_Y(D_2) + \epsilon \quad (49)$$

Then,

$$C_{GK}(X, Y; D_1, D_2) = \sup R_0 \in \mathcal{R}_{GK}(D_1, D_2) \quad (50)$$

Again, observe that, if every point on the intersection of the planes  $R_0 + R_1 = R_X(D_1)$  and  $R_0 + R_2 = R_Y(D_2)$  satisfies (5) with equality for some joint density  $P(X, Y, \hat{X}, \hat{Y}, U)$ , then the above definition can be simplified by setting  $\epsilon = 0$ . Hereafter, we will assume that this condition holds, noting that the results can be extended to the general case, similar to arguments in Appendix A.

### B. Single Letter Characterization of $C_{GK}(X, Y; D_1, D_2)$

We provide an information theoretic characterization for  $C_{GK}(X, Y; D_1, D_2)$  in the following theorem.

*Theorem 3: A single letter characterization of  $C_{GK}(X, Y; D_1, D_2)$  is given by:*

$$C_{GK}(X, Y; D_1, D_2) = \sup I(X, Y; U) \quad (51)$$

where the supremum is over all joint densities  $(X, Y, \hat{X}, \hat{Y}, U)$  such that the following Markov conditions hold:

$$\begin{aligned} Y &\leftrightarrow X \leftrightarrow U & X &\leftrightarrow Y \leftrightarrow U \\ X &\leftrightarrow \hat{X} \leftrightarrow U & Y &\leftrightarrow \hat{Y} \leftrightarrow U \end{aligned} \quad (52)$$

where  $P(\hat{X}|X) \in \mathcal{P}_{D_1}^X$  and  $P(\hat{Y}|Y) \in \mathcal{P}_{D_2}^Y$ , are any rate-distortion optimal encoders at  $D_1$  and  $D_2$ , respectively.

*Proof:* The proof follows in very similar lines to the proof of Theorem 1. The original Gray-Wyner characterization is, in fact, sufficient in this case. We first assume that there are unique encoders  $P(\hat{X}|X)$  and  $P(\hat{Y}|Y)$ , that achieve  $R_X(D_1)$  and  $R_Y(D_2)$ , respectively. The proof extends directly to the case of multiple rate-distortion optimal encoders.

We are interested in characterizing the points in  $\mathcal{R}_{GW}(D_1, D_2)$  which lie on both the planes  $R_0 + R_1 = R_X(D_1)$  and  $R_0 + R_2 = R_Y(D_2)$ . Therefore we have the following series of inequalities:

$$\begin{aligned} R_X(D_1) &= R_0 + R_1 \\ &\geq I(X, Y; U) + I(X; \hat{X}|U) \\ &= I(X; \hat{X}, U) + I(Y; U|X) \\ &\geq I(X; \hat{X}) \geq R_X(D_1) \end{aligned} \quad (53)$$

Writing similar inequality relations for  $Y$  and following the same arguments as in Theorem 1, it follows that for all

joint densities satisfying (52) and for which  $P(\hat{X}|X) \in \mathcal{P}_{D_1}^X$  and  $P(\hat{Y}|Y) \in \mathcal{P}_{D_2}^Y$ , there exists at least one point in  $\mathcal{R}_{GW}(D_1, D_2)$  which satisfies both  $R_0 + R_1 = R_X(D_1)$  and  $R_0 + R_2 = R_Y(D_2)$  and for which  $R_0 = I(X, Y; U)$ . This proves the theorem. ■

*Corollary 2:  $C_{GK}(X, Y; D_1, D_2) \leq C_{GK}(X, Y)$*

*Proof:* This corollary follows directly from Theorem 3 as conditions in (16) are a subset of the conditions in (52). ■

It is easy to show that if the random variables  $(X, Y)$  are jointly Gaussian with a correlation coefficient  $\rho < 1$ , then  $C_{GK}(X, Y) = 0$ . Hence from Corollary 2, it follows that, for jointly Gaussian random variables with correlation coefficient strictly less than 1,  $C_{GK}(X, Y; D_1, D_2) = 0 \forall D_1, D_2$ , under any distortion measure. It is well known that  $C_{GK}(X, Y)$  is typically very small and is non-zero only when the ergodic decomposition of the joint distribution leads to non-trivial subsets of the alphabet space. In the general setting, as  $C_{GK}(X, Y; D_1, D_2) \leq C_{GK}(X, Y)$ , it would seem that Theorem 3 has very limited practical significance. However, in [16] we showed that  $C_{GK}(X, Y; D_1, D_2)$  plays a central role in scalable coding of sources that are not successively refinable. Further implications of this result will be studied in more detail as part of future work.

## V. OPTIMAL TRANSMIT-RECEIVE RATE TRADEOFF IN GRAY-WYNER NETWORK

### A. Motivation

It is well known that the two definitions of CI, due to Wyner and Gács-Körner, can be characterized using the Gray-Wyner region and the corresponding operating points are two boundary points of the region. Several approaches have been proposed to provide further insight into the underlying connections between them [5], [6], [9], [17]. However, to the best of our knowledge, no prior work has identified an operationally significant contour of points on the Gray-Wyner region, which connects the two operating points. In this section we derive and analyze such a contour of points on the boundary, obtained by trading-off a particular definition of transmit rate and receive rate, which passes through both the operating points of Wyner and Gács-Körner. This tradeoff provides a generic framework to understand the underlying principles of shared information. We note in passing that Paul Cuff characterized a tradeoff between Wyner's common information and the mutual information in [18], while studying the amount of common randomness required to generate correlated random variables, with communication constraints. This tradeoff is similar in spirit to the tradeoff studied in this paper, although very different in details.

We define the total transmit rate for the Gray-Wyner network as  $R_t = R_0 + R_1 + R_2$  and the total receive rate as  $R_r = 2R_0 + R_1 + R_2$ . Specifically, we show that the contour traced on the Gray-Wyner region boundary when we trade  $R_t = R_0 + R_1 + R_2$  for  $R_r = 2R_0 + R_1 + R_2$ , passes through both the operating points of Wyner and Gács-Körner for any distortion pair  $(D_1, D_2)$ .

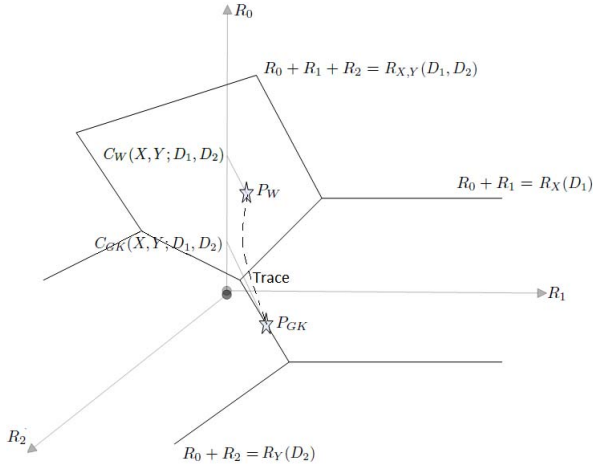


Fig. 3.  $P_W$  and  $P_{GK}$  in the Gray-Wyner region. Observe that the transmit contour and the receive contour coincide in between  $P_W$  and  $P_{GK}$ .

### B. Relating the Two notions of CI

Let  $P_W$  and  $P_{GK}$  denote the respective operating points in  $\mathcal{R}_{GW}(D_1, D_2)$ , corresponding to the lossy definitions of Wyner and Gács-Körner CI ( $C_W(X, Y; D_1, D_2)$  and  $C_{GK}(X, Y; D_1, D_2)$ ). Hereafter, the dependence on the distortion constraints will be implicit for notational convenience.  $P_W$  and  $P_{GK}$  are shown in Fig. 3.

We define the following two contours in the Gray-Wyner region. We define the transmit contour as the set of points on the boundary of the Gray-Wyner region obtained by minimizing the total receive rate ( $R_r$ ) at different total transmit rates ( $R_t$ ), i.e., the transmit contour is the trace of operating points obtained when the receive rate is minimized subject to a constraint on the transmit rate. Similarly, we define the receive contour as the trace of points on the Gray-Wyner region obtained by minimizing the transmit rate ( $R_t$ ) for each ( $R_r$ ).

*Claim:* The transmit contour coincides with the receive contour between  $P_W$  and  $P_{GK}$ .

*Proof:*  $\mathcal{R}_{GW}$  is a convex region. Hence the set of achievable rate pairs for  $(R_t, R_r) = (R_0 + R_1 + R_2, 2R_0 + R_1 + R_2)$  is convex. We have  $R_t \geq R_{X,Y}(D_1, D_2)$  and  $R_r \geq R_X(D_1) + R_Y(D_2)$ . Note that when  $R_t = R_{X,Y}(D_1, D_2)$ ,  $\min R_r = R_{X,Y}(D_1, D_2) + C_W(X, Y; D_1, D_2)$  and is achieved at  $P_W$ . Similarly when  $R_r = R_X(D_1) + R_Y(D_2)$ ,  $\min R_t = R_X(D_1) + R_Y(D_2) - C_{GK}(X, Y; D_1, D_2)$ , which is achieved at  $P_{GK}$ . Figure 4 depicts the trade-off between  $R_t$  and  $R_r$ . Hence, it follows from the convexity of  $(R_t, R_r)$  region that for every transmit rate  $R_t \geq R_{X,Y}(D_1, D_2)$ , there exists a receive rate  $R_r \leq R_X(D_1) + R_Y(D_2)$ , such that, the corresponding operating points on the transmit and the receive contours respectively coincide. Hence, it follows that the transmit and the receive contours coincide in between  $P_W$  and  $P_{GK}$ . ■

This new relation between the two notions of CI brings them both under a common framework. Gács and Körner's operating point can now be stated as the *minimum* shared rate (similar to Wyner's definition), at a sufficiently large sum rate.

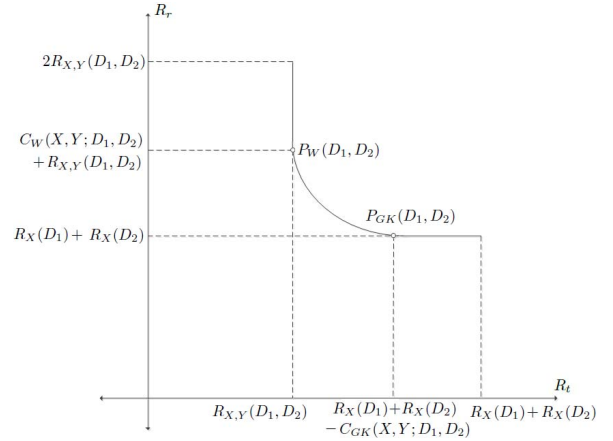


Fig. 4. Tradeoff between  $R_r$  and  $R_t$ . Observe that, when  $R_t = R_{X,Y}(D_1, D_2)$ , minimum  $R_r$  is equal to  $R_{X,Y}(D_1, D_2) + C_W(X, Y; D_1, D_2)$  and when  $R_r = R_X(D_1) + R_Y(D_2)$ , minimum  $R_t = R_X(D_1) + R_Y(D_2) - C_{GK}(X, Y; D_1, D_2)$ .

Likewise, Wyner's CI can be defined as the *maximum* shared rate (similar to Gács and Körner's definition), at a sufficiently large receive rate. We will make these arguments more precise in Section V-E when we derive alternate characterizations in the lossless framework for each notion in terms of the objective function of the other.

We note that operating point corresponding to Gács-Körner CI is always unique, for all  $(D_1, D_2)$ , as it lies on the intersection of two planes. However, the operating point corresponding to Wyner's CI may not be unique, i.e., there could exist source and distortion pairs for which, the minimum shared rate on the Pangloss plane could be achieved at multiple points in the GW region. In fact, it follows directly from the convexity of the GW region that if there are two operating points corresponding to Wyner's CI, then all points in between them also achieve minimum shared rate and lie on the Pangloss plane. For such sources and distortion pairs, the trade-off between the transmit and receive rates on the GW network leads to a surface of operating points, instead of a contour. Nevertheless, this surface always intersects both  $P_W$  and  $P_{GK}$ .

### C. Single Letter Characterization of the Tradeoff

The tradeoff between the transmit and the receive rates in the Gray-Wyner network not only plays a crucial role in providing theoretical insight into the workings of the two notions of CI, but also has implications in several practical scenarios such as fusion coding and selective retrieval of correlated sources in a database [19] and dispersive information routing of correlated sources [20], as described in [4]. It is therefore of interest to derive a single letter information theoretic characterization for this tradeoff. Although we were unable to derive a complete characterization for general distortions, we derive a single letter complete characterization for the lossless setting here. Hence, we focus only on the lossless setting for the rest of the paper.

To gain insight into this tradeoff, we characterize two curves which are rotated/transformed versions of each other. The first curve, denoted by  $C(X, Y; R')$ , plots the minimum

shared rate,  $R_0$ , at a transmit rate of  $H(X, Y) + R'$  and the second, denoted by  $K(X, Y; R'')$ , is the maximum  $R_0$  at a receive rate of  $H(X) + H(Y) + R''$ . It is easy to see that the transmit-receive rate tradeoff can be derived directly from these quantities. We note that the quantities  $C(X, Y; R')$  and  $K(X, Y; R'')$  are in fact generalizations of Wyner and Gács-Körner (lossless) definitions of CI to the excess sum transmit rate and receive rate regimes, respectively. Using their properties, we will also derive alternate characterizations for the two notions of lossless CI under a unified framework in Section V-D.

We define the quantity  $C(X, Y; R') \forall R' \in [0, I(X, Y)]$  as:

$$C(X, Y; R') = \inf R_0 : (R_0, R_1, R_2) \in \mathcal{R}_{GW} \quad (54)$$

satisfying,

$$R_0 + R_1 + R_2 = H(X, Y) + R' \quad (55)$$

Similarly, we define the quantity  $K(X, Y; R'') \forall R'' \in [0, H(X, Y) - I(X, Y)]$  as:

$$K(X, Y; R'') = \sup R_0 : (R_0, R_1, R_2) \in \mathcal{R}_{GW} \quad (56)$$

satisfying,

$$2R_0 + R_1 + R_2 = H(X) + H(Y) + R'' \quad (57)$$

Note that we restrict the ranges for  $R'$  and  $R''$  to the ranges of practical interest, as operating at  $R' > I(X, Y)$  or  $R'' > H(X, Y) - I(X, Y)$  is suboptimal and uninteresting. The following Theorem provides information theoretic characterizations for  $C(X, Y; R')$  and  $K(X, Y; R'')$ .

*Theorem 4:* (i) For any excess sum transmit rate  $R' \in [0, I(X, Y)]$ :

$$C(X, Y; R') = \min I(X, Y; U) \quad (58)$$

where the minimization is over all  $U$  jointly distributed with  $(X, Y)$  such that:

$$I(X; Y|U) = R' \quad (59)$$

We denote the operating point in  $\mathcal{R}_{GW}$  corresponding to the minimum by  $P_{C(X, Y)}(R')$ .

(ii) For any excess reception rate  $R'' \in [0, H(X, Y) - I(X, Y)]$ :

$$K(X, Y; R'') = \max I(X, Y; W) \quad (60)$$

where the maximization is over all  $W$  jointly distributed with  $(X, Y)$  such that:

$$I(X; W|Y) + I(Y; W|X) = R'' \quad (61)$$

We denote the operating point in  $\mathcal{R}_{GW}$  corresponding to the maximum by  $P_{K(X, Y)}(R'')$ .

*Proof:* We prove part (i) of the theorem for  $C(X, Y; R')$ . The proof of (ii) for  $K(X, Y; R'')$  follows similar lines.

*Achievability:* Let  $U$  be jointly distributed with  $(X, Y)$  such that  $I(X; Y|U) = R'$ . It leads to a point in the Gray-Wyner region with  $(R_0, R_1, R_2) = (I(X, Y; U), H(X|U), H(Y|U))$ . On substituting in (55) we have:

$$\begin{aligned} R_0 + R_1 + R_2 &= I(X, Y; U) + H(X|U) + H(Y|U) \\ &= H(X, Y) + I(X; Y|U) \\ &= H(X, Y) + R' \end{aligned} \quad (62)$$

$$= H(X, Y) + R' \quad (63)$$

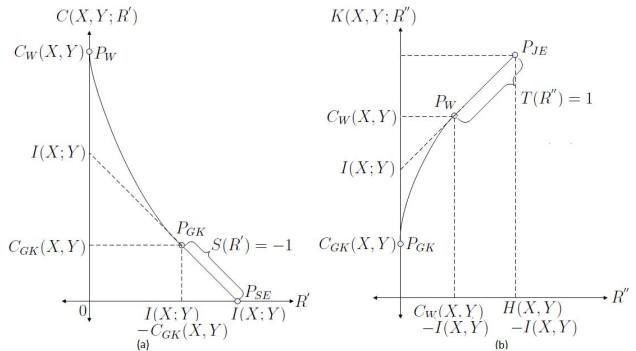


Fig. 5. (a) Typical transmit tradeoff curve -  $C(X, Y; R')$ . (b) Typical receive tradeoff curve -  $K(X, Y; R'')$ .

Note that the existence of a  $U$  that achieves the minimum in (58) follows from Theorem 4.4 (A) in [2]. This allows us to replace the infimum in the definition of  $C(X, Y; R')$  with a minimum in (58).

*Converse:* We know from the converse to the Gray-Wyner region that every point in  $\mathcal{R}_{GW}$  is achieved by some random variable  $U$  jointly distributed with  $(X, Y)$ . We need to determine the condition on  $U$  for (55) to hold. On substituting  $(R_0, R_1, R_2) = (I(X, Y; U), H(X|U), H(Y|U))$  in (55), we get the condition to be (59), proving the converse. ■

Note that the cardinality of  $\mathcal{U}$  can be restricted to  $|\mathcal{U}| \leq |\mathcal{X}||\mathcal{Y}| + 1$  using Theorem 4.4 in [2]. Also note that when the transmit rate is  $H(X, Y) + R'$ , the minimum receive rate is  $H(X, Y) + R' + C(X, Y; R')$ . Similarly, when the receive rate is  $H(X) + H(Y) + R''$ , the minimum transmit rate is  $H(X) + H(Y) + R'' - K(X, Y; R'')$ . Hence the quantities  $C(X, Y; R')$  and  $K(X, Y; R'')$  are just rotated/transformed versions of the transmit versus receive rate tradeoff curve.

We refer to plots of  $C(X, Y; R')$  and  $K(X, Y; R'')$  versus  $R'$  and  $R''$  as the 'transmit tradeoff curve' and the 'receive tradeoff curve', respectively. Observe that in both cases, as we increase  $R'$  (or  $R''$ ) we obtain parallel cross-sections of the Gray-Wyner region and the set of operating points  $P_{C(X, Y)}(R')$  and  $P_{K(X, Y)}(R'')$ , trace contours on the boundary of the region. These two contours are precisely the transmit and the receive contours defined in section V-B. Note the difference between the contours and their respective tradeoff curves. The contours are defined in a 3-D space and lie on the boundary of  $\mathcal{R}_{GW}$ . In general each of the contours may not even lie on a single plane. However, the tradeoff curves are a projection of the respective contours on to a 2-D plane.

#### D. Properties of the Tradeoff Curve

In this section, we focus on the quantities  $C(X, Y; R')$  and  $K(X, Y; R'')$  and analyze some important properties, which allow us to provide alternate characterizations for the two notions of CI. As most of the proofs for  $K(X, Y; R'')$  are very similar to their  $C(X, Y; R')$  counterparts, we only prove the properties for  $C(X, Y; R')$ . We plot typical transmit and receive tradeoff curves in Figure 5 to illustrate the discussion.

Consider the transmit tradeoff curve. At  $R' = 0$  we get the operating point corresponding to Wyner CI where the

minimum shared information is given by  $C_W(X, Y)$ . This point is denoted by  $P_W$  in Figure 5. Next observe that at  $R_0 = 0$ , for lossless reconstruction of  $X$  and  $Y$ , we need,  $R_1 \geq H(X)$  and  $R_2 \geq H(Y)$ . Therefore at an excess sum transmit rate  $R' = H(X) + H(Y) - H(X, Y) = I(X; Y)$ , the shared rate vanishes, or,  $C(X, Y; I(X; Y)) = 0$ . We call this point - ‘separate encoding’ and denote it by  $P_{SE}$  in the figure. It is also obvious that any  $U$  independent of  $(X, Y)$  achieves this minimum  $R_0$  for  $R' = I(X; Y)$ .

*Lemma 2 (Convexity):*  $C(X, Y; R')$  is convex for  $R' \in [0, I(X; Y)]$  and  $K(X, Y; R'')$  is concave for  $R'' \in [0, H(X, Y) - I(X; Y)]$ .

*Proof:* The proof follows directly from the convexity of the Gray-Wyner region. ■

*Lemma 3 (Monotonicity 3):*  $C(X, Y; R')$  is strictly monotone decreasing  $\forall R' \in [0, I(X; Y)]$  and  $K(X, Y; R'')$  is strictly monotone increasing  $\forall R'' \in [0, H(X, Y) - I(X; Y)]$

*Proof:* It is clear from the achievability results of Gray-Wyner that if a point  $(r_0, r_1, r_2) \in \mathcal{R}_{GW}$ , then all points  $\{(R_0, R_1, R_2) : R_0 \geq r_0, R_1 \geq r_1, R_2 \geq r_2\} \in \mathcal{R}_{GW}$ . Let  $C(X, Y; R') = r_0$ , for some excess transmission rate  $R'$  and let the corresponding operating point in  $\mathcal{R}_{GW}$  be  $(r_0, r_1, r_2)$ . Hence for any  $\Delta > 0$ , the point  $(r_0, r_1 + \Delta, r_2) \in \mathcal{R}_{GW}$  and satisfies  $R_0 + R_1 + R_2 = R + \Delta$ . Therefore,

$$C(X, Y; R' + \Delta) = \inf R_0 : \{R_0 + R_1 + R_2 = R + \Delta\} \leq r_0 \quad (64)$$

Hence,  $C(X, Y; R')$  is non-increasing. Then it follows from convexity that  $C(X, Y; R')$  is either a constant or is strictly monotone decreasing. Lemma 4 below eliminates the possibility of a constant, proving this lemma. ■

At all  $R'$  where  $C(X, Y; R')$  is differentiable, we denote the slope by  $S(R')$ . At non-differentiable points, we denote by  $S^-(R')$  and  $S^+(R')$  the left and right derivatives, respectively. Similarly the slope, left derivative and right derivatives of  $K(X, Y; R'')$  are denoted by  $T(R'')$ ,  $T^-(R'')$  and  $T^+(R'')$ , respectively.

*Lemma 2:* The slope of  $C(X, Y; R')$ ,  $S(R') \leq -1 \forall R' \in [0, I(X; Y)]$  where the curve is differentiable. At non-differentiable points, we have  $S^-(R') < S^+(R') \leq -1$ . Similarly we have,  $T(R'') \geq 1 \forall R'' \in [0, H(X, Y) - I(X; Y)]$  and  $T^-(R'') > T^+(R'') \geq 1$

*Proof:* Note that it is sufficient for us to show that  $S^-(I(X; Y)) \leq -1$ . Then it directly follows from convexity that  $S(R') \leq -1$  at all differentiable points and  $S^-(R') < S^+(R') \leq -1$  at all non-differentiable points. Consider  $\Delta > 0$ , and fix the shared information rate to be  $R_0 = \Delta$ . From the converse of the source coding theorem for lossless reconstruction, we have:

$$\begin{aligned} R_0 + R_1 &= \Delta + R_1 \geq H(X) \\ R_0 + R_2 &= \Delta + R_2 \geq H(Y) \end{aligned} \quad (65)$$

The above inequalities imply  $R_0 + R_1 + R_2 \geq H(X, Y) + I(X; Y) - \Delta$ . Therefore the point on the transmit tradeoff curve with  $C(X, Y; R') = \Delta$  has  $R' \geq I(X; Y) - \Delta$ . Hence  $S^-(I(X; Y)) \leq -1$  proving the Lemma. ■

*Remark 3:* Observe that the above proofs do not rely on the lossless definitions of  $C(X, Y; R')$  and  $K(X, Y; R'')$ , but leverage only on the convexity of the lossless Gray-Wyner region. It is well known that the lossy Gray-Wyner region is also convex in the rates, for all distortion pairs. Consequently, all the three lemmas (2, 3 and 4) can be easily extended to the lossy counterparts of  $C(X, Y; R')$  and  $K(X, Y; R'')$ . We omit the details of the proof here to avoid repetition.

### E. Alternate Characterizations for $C_{GK}(X, Y)$ and $C_W(X, Y)$

In this section, we provide alternate characterizations for  $C_{GK}(X, Y)$  and  $C_W(X, Y)$  in terms of  $C(X, Y; R')$  and  $C(X, Y; R'')$ , respectively.

*Theorem 5:* An alternate characterization for the Gács-Körner CI is:

$$C_{GK}(X, Y) = \sup_{R': S^+(R') = -1} C(X, Y; R') \quad (66)$$

If there exists no  $R'$  for which  $S^+(R') = -1$ , then,  $C_{GK}(X, Y) = 0$ . Similarly, an alternate characterization for Wyner’s CI is :

$$C_W(X, Y) = \inf_{R'': T^+(R'') = 1} K(X, Y; R'') \quad (67)$$

If there exists no  $R''$  for which  $T^+(R'') = 1$ , then,  $C_W(X, Y) = H(X, Y)$ . Note that  $C_{GK}(X, Y)$  corresponds to that excess sum transmit rate where the region of  $C(X, Y; R')$  with slope  $< -1$  meets the region with slope equal to  $-1$ , and  $C_W(X, Y)$  corresponds to that excess receive rate where the region of  $K(X, Y; R'')$  with slope  $> 1$  meets the region with slope equal to 1.

*Proof:* We first assume that there exists some  $R^* \in [0, I(X; Y)]$ , which is the minimum rate at which  $S^+(R^*) = -1$ . We need to show that  $C(X, Y; R^*) = C_{GK}(X, Y)$ . We denote this point by  $P_{GK}$  in the figure. Let  $\tilde{R}$  be such that  $R^* \leq \tilde{R} < I(X; Y)$  and let  $\tilde{U}$  be the random variable which achieves the minimum shared information rate at  $\tilde{R}$  in Theorem 4. Then it follows from Lemmas 2 and 4 that  $S^+(\tilde{R}) = -1$ . Then the point in the GW region corresponding to  $\tilde{U}$  satisfies the following two conditions:

$$\begin{aligned} R_0 &= I(X, Y) - \tilde{R} \\ R_0 + R_1 + R_2 &= H(X, Y) + \tilde{R} \end{aligned} \quad (68)$$

Adding the two equations, we have  $2R_0 + R_1 + R_2 = H(X) + H(Y)$ , which implies that  $R_0 + R_1 = H(X)$  and  $R_0 + R_2 = H(Y)$ . Therefore, the point corresponding to  $\tilde{U}$  satisfies Gács-Körner constraints (14). Hence, it follows that, any  $\tilde{R}$  such that  $S^+(\tilde{R}) = -1$  leads to an operating point in the GW region which satisfies Gács-Körner constraints.

Next, we need to show the converse. Consider any point in the GW region satisfying Gács-Körner constraints. It can be written as,

$$\begin{aligned} R_0 &= I(X; Y) - \tilde{R} \\ R_1 &= H(X) - (I(X; Y) - \tilde{R}) \\ R_2 &= H(Y) - (I(X; Y) - \tilde{R}) \end{aligned} \quad (69)$$

for some  $C_{GK}(X, Y) \leq \tilde{R} \leq I(X; Y)$ . On summing the three equations, we have  $R_0 + R_1 + R_2 = H(X, Y) + \tilde{R}$ . It then follows from the convexity of  $C(X, Y; R')$  that  $S^+(\tilde{R}) = -1$ . Therefore, we have,

$$\begin{aligned} C(X, Y; R^*) &= I(X; Y) - R^* \\ &= I(X; Y) - \min_{\tilde{R}: S^+(\tilde{R}) = -1} \tilde{R} \\ &= \max \left\{ R_0 : (R_0, R_1, R_2) \in \mathcal{R}_{GW} \right. \\ &\quad \left. R_0 + R_1 = H(X), R_0 + R_2 = H(Y) \right\} \\ &= C_{GK}(X, Y) \end{aligned} \quad (70)$$

proving the first part of the theorem. However, if there exists no  $R' \in [0; I(X; Y)]$  for which  $S^+(R') = -1$ , it implies that  $\forall R' \in [0; I(X; Y))$ ,  $C(X, Y; R') > I(X; Y)$ . Therefore (14) is not satisfied with equality for any  $R' \in [0; I(X; Y))$ . Hence  $C_{GK}(X, Y) = 0$ . ■

Note that the above characterizations for  $C_{GK}(X, Y)$  and  $C_W(X, Y)$  are of fundamentally different nature from their original characterizations and provide important insights into the understanding of shared information. Moreover, from a practical standpoint, these characterizations also play a role in finding the minimum communication cost for networks when the cost of transmission on each link is a non-linear function of the rate as illustrated in [4].

## VI. CONCLUSION

In this paper we derived single letter information theoretic characterizations for the lossy generalizations of the two most prevalent notions of CI due to Wyner and due to Gács and Körner. These generalizations allow us to extend the theoretical interpretation underlying their original definitions to sources with infinite entropy (eg. continuous random variables). We use these information theoretic characterizations to derive the CI of bivariate Gaussian random variables. We finally showed that the operating points associated with the two notions of CI arise as extreme special cases of a broader framework, that involves the tradeoff between the total transmit versus the receive rate in the Gray-Wyner network. For the lossless setting, single letter information theoretic characterization for the tradeoff curve was established. Using the properties of the tradeoff curve, alternate characterizations under a common framework were derived for the two notions of CI.

## APPENDIX A

### PROOF OF THEOREM 1 FOR THE GENERAL SETTING

In this appendix, we extend the proof of Theorem 1 for general sources and distortion measures. Here, we do *not* assume that every point on the intersection of the Gray-Wyner region and the Pangloss plane satisfies (5) with equality for some joint density  $P(X, Y, U, \hat{X}, \hat{Y})$ . We note that an equivalent definition of Wyner's lossy CI is given by the following. For any  $\epsilon > 0$ , let  $R_0^{\min}(D_1, D_2, \epsilon)$  be defined as:

$$R_0^{\min}(D_1, D_2, \epsilon) = \inf R_0 \quad (71)$$

over all points  $(R_0, R_1, R_2)$  satisfying:

$$\begin{aligned} (R_0, R_1, R_2) &\in \mathcal{R}_{GW}(D_1, D_2) \\ R_0 + R_1 + R_2 &\leq R_{X,Y}(D_1, D_2) + \epsilon \end{aligned} \quad (72)$$

Then,

$$C_W(X, Y; D_1, D_2) = \lim_{\epsilon \rightarrow 0} R_0^{\min}(D_1, D_2, \epsilon) \quad (73)$$

In the following Lemma, we derive upper and lower bounds to  $C_W(X, Y; D_1, D_2)$  in terms of  $\epsilon$ .

*Lemma 3: Let  $\epsilon > 0$  be given. Then, an upper bound to  $C_W(X, Y; D_1, D_2)$  is:*

$$C_W(X, Y; D_1, D_2) \leq \inf I(X, Y; U) \quad (74)$$

where the infimum is over all joint densities  $P(X, Y, \hat{X}, \hat{Y}, U)$  satisfying:

$$\begin{aligned} I(X, Y; \hat{X}, \hat{Y}) &\leq R_{X,Y}(D_1, D_2) + \epsilon \\ I(X, Y; U | \hat{X}, \hat{Y}) &= 0 \\ I(\hat{X}; \hat{Y} | U) &= 0 \\ E(d_X(X, \hat{X})) &\geq D_1 \\ E(d_Y(Y, \hat{Y})) &\geq D_2 \end{aligned} \quad (75)$$

We denote this upper bound by  $C_W^{UB}(D_1, D_2, \epsilon)$ .

A lower bound to  $C_W(X, Y; D_1, D_2)$  is:

$$C_W(X, Y; D_1, D_2) \geq \inf I(X, Y; U) \quad (76)$$

where the infimum is over all joint densities  $P(X, Y, \hat{X}, \hat{Y}, U)$  satisfying:

$$\begin{aligned} I(X, Y; \hat{X}, \hat{Y}) &\leq R_{X,Y}(D_1, D_2) + \epsilon \\ I(X, Y; U | \hat{X}, \hat{Y}) &\leq \epsilon \\ I(\hat{X}; \hat{Y} | U) &\leq \epsilon \\ E(d_X(X, \hat{X})) &\geq D_1 \\ E(d_Y(Y, \hat{Y})) &\geq D_2 \end{aligned} \quad (77)$$

We denote this lower bound by  $C_W^{LB}(D_1, D_2, \epsilon)$ .

*Proof:* The proof follows using very similar arguments to that in the proof of Theorem 1. Hence, we omit the details here to avoid repetition. ■

Observe that the proof of Theorem 1 for the general setting follows once we show that  $C_W^{UB}(D_1, D_2, \epsilon)$  and  $C_W^{LB}(D_1, D_2, \epsilon)$  are continuous at  $\epsilon = 0$ . The following Lemma sheds light on the the continuity of these quantities at  $\epsilon = 0$ .

*Lemma 4: Let  $(D_1, D_2)$  be a pair of distortions at which there exists at least one distribution  $P(\hat{X}, \hat{Y} | X, Y)$  that is RD-optimal in achieving  $R_{X,Y}(D_1, D_2)$ . Then:*

$$C_W(X, Y; D_1, D_2) = C_W^{LB}(D_1, D_2, 0) = C_W^{UB}(D_1, D_2, 0) \quad (78)$$

However, if there exists no RD-optimal distribution, i.e., there only exist distributions that can infinitesimally approach  $R_{X,Y}(D_1, D_2)$ , then:

$$\begin{aligned} C_W(X, Y; D_1, D_2) &= \lim_{\epsilon \rightarrow 0} C_W^{LB}(D_1, D_2, \epsilon) \\ &= \lim_{\epsilon \rightarrow 0} C_W^{UB}(D_1, D_2, \epsilon) \end{aligned} \quad (79)$$

*Proof:* To prove this Lemma, we employ techniques very similar to the ones used by Wyner in [2]. We further

restrict ourselves to discrete random variables for simplicity. However, the arguments can be easily extended to well-behaved continuous random variables and distortion measures using standard techniques.

We first show that the quantity  $C_W^{LB}(D_1, D_2, \epsilon)$  is a convex function of  $\epsilon$  for all  $\epsilon > 0$ . Towards proving this, define a generalized version of  $C_W^{LB}(D_1, D_2, \epsilon)$  as:

$$C_W^{LB}(D_1, D_2, \epsilon_1, \epsilon_2, \epsilon_3) = \inf I(X, Y; U) \quad (80)$$

where the infimum is over all joint densities  $P(X, Y, \hat{X}, \hat{Y}, U)$  satisfying:

$$\begin{aligned} I(X, Y; \hat{X}, \hat{Y}) &\leq R_{X,Y}(D_1, D_2) + \epsilon_1 \\ I(X, Y; U|\hat{X}, \hat{Y}) &\leq \epsilon_2 \\ I(\hat{X}; \hat{Y}|U) &\leq \epsilon_3 \\ E(d_X(X, \hat{X})) &\geq D_1 \\ E(d_Y(Y, \hat{Y})) &\geq D_2 \end{aligned} \quad (81)$$

Particularly, we show that  $C_W^{LB}(D_1, D_2, \epsilon_1, \epsilon_2, \epsilon_3)$  is convex with respect to  $\epsilon_i$  for any fixed values of  $\epsilon_j$  and  $\epsilon_k$ ,  $i, j, k \in \{1, 2, 3\}$ .

Let  $\epsilon_2 > 0$  and  $\epsilon_3 > 0$  be fixed. Let  $\epsilon_{11} > 0$  and  $\epsilon_{12} > 0$  be two values for  $\epsilon_1$  and let the corresponding optimizing distributions for (80) be  $P_1(X, Y, \hat{X}, \hat{Y}, U)$  and  $P_2(X, Y, \hat{X}, \hat{Y}, U)$ , respectively. Now consider  $\epsilon_1 = \theta\epsilon_{11} + (1 - \theta)\epsilon_{12}$ , for some  $0 < \theta < 1$ . It is easy to check that the joint distribution that takes value  $P_1$  with probability  $\theta$  and value  $P_2$  with probability  $1 - \theta$ , denoted hereafter by  $P_\theta$ , satisfies all the constraints in (81) for  $\epsilon_1, \epsilon_2$  and  $\epsilon_3$ . Next, consider the following series of inequalities:

$$\begin{aligned} C_W^{LB}(D_1, D_2, \epsilon_1, \epsilon_2, \epsilon_3) &\leq I^{P_\theta}(X, Y; U) \\ &= \theta I^{P_1}(X, Y; U) + (1 - \theta) I^{P_2}(X, Y; U) \\ &= \theta C_W^{LB}(D_1, D_2, \epsilon_{11}, \epsilon_2, \epsilon_3) \\ &\quad + (1 - \theta) C_W^{LB}(D_1, D_2, \epsilon_{12}, \epsilon_2, \epsilon_3) \end{aligned}$$

where  $I^P(\cdot, \cdot)$  denotes the mutual information with respect to joint density  $P$ . This proves convexity of  $C_W^{LB}(D_1, D_2, \epsilon_1, \epsilon_2, \epsilon_3)$  with respect to  $\epsilon_1$  for fixed  $\epsilon_2$  and  $\epsilon_3$ . Similar arguments lead to the conclusion that  $C_W^{LB}(D_1, D_2, \epsilon_1, \epsilon_2, \epsilon_3)$  is convex with respect to  $(\epsilon_1, \epsilon_2, \epsilon_3)$  when  $\epsilon_1 > 0$ ,  $\epsilon_2 > 0$  and  $\epsilon_3 > 0$ . Hence,  $C_W^{LB}(D_1, D_2, \epsilon)$  and  $C_W^{UB}(D_1, D_2, \epsilon)$  are convex and continuous for all  $\epsilon > 0$ .

To prove that  $C_W^{LB}(D_1, D_2, \epsilon_1, \epsilon_2, \epsilon_3)$  is continuous at the origin, we first consider continuity with respect to  $\epsilon_2$  for fixed  $\epsilon_1$  and  $\epsilon_3$ . Let  $\epsilon_1, \epsilon_2, \epsilon_3 > 0$  and let  $P(X, Y, \hat{X}, \hat{Y}, U)$  be the joint density that achieves the optimum for  $C_W^{LB}(D_1, D_2, \epsilon_1, \epsilon_2, \epsilon_3)$ . It is sufficient for us to prove that there exists a joint density  $Q(X, Y, \hat{X}, \hat{Y}, U)$  that satisfies (81) with  $\epsilon_2 = 0$ , and for which  $I^Q(X, Y; U)$  is within  $\delta(\epsilon_2)$  from  $C_W^{LB}(D_1, D_2, \epsilon_1, \epsilon_2, \epsilon_3)$ , for some  $\delta(\epsilon_2) \rightarrow 0$  as  $\epsilon_2 \rightarrow 0$ . We construct the joint density  $Q(\cdot)$  as follows:

$$Q(X, Y, \hat{X}, \hat{Y}, U) = P(\hat{X}, \hat{Y})P(X, Y|\hat{X}, \hat{Y})P(U|\hat{X}, \hat{Y})$$

Observe that all the conditions in (81) with  $\epsilon_2 = 0$  are satisfied by  $Q(\cdot)$ , as  $I^Q(X, Y; U|\hat{X}, \hat{Y}) = 0$ . We need to show that

$|I^P(X, Y; U) - I^Q(X, Y; U)| \leq \delta(\epsilon_2)$  for some  $\delta(\epsilon_2) \rightarrow 0$  as  $\epsilon_2 \rightarrow 0$ . Towards proving this result, we have:

$$\begin{aligned} \epsilon_2 &= I^P(X, Y; U|\hat{X}, \hat{Y}) \\ &\geq^{(a)} \sup |P(X, Y, U|\hat{X}, \hat{Y}) - Q(X, Y, U|\hat{X}, \hat{Y})| \end{aligned}$$

where (a) follows from Pinsker's inequality [21] and the supremum is over all possible subsets of alphabets of  $(X, Y, U, \hat{X}, \hat{Y})$ . The above inequalities state that the joint densities  $P(X, Y, \hat{X}, \hat{Y}, U)$  and  $Q(X, Y, \hat{X}, \hat{Y}, U)$  have a total variation smaller than  $\epsilon_2$ . Therefore, as  $\epsilon_2 \rightarrow 0$ ,  $P(\cdot) \rightarrow Q(\cdot)$ . As conditional entropy is continuous in the total variation distance of the corresponding random variables, there exists a  $\delta(\epsilon_2)$  such that  $|I^P(X, Y; U) - I^Q(X, Y; U)| \leq \delta(\epsilon_2)$  and  $\delta(\epsilon_2) \rightarrow 0$  as  $\epsilon_2 \rightarrow 0$ . This proves that  $\lim_{\epsilon_2 \rightarrow 0} C_W^{LB}(D_1, D_2, \epsilon_1, \epsilon_2, \epsilon_3) = C_W^{LB}(D_1, D_2, \epsilon_1, 0, \epsilon_3)$ . The proof for continuity with respect to  $\epsilon_3$  at origin follows in very similar lines to the proof of [2, Th. 4.4]. Hence, we omit the details here. This leads to the conclusion that:

$$\begin{aligned} C_W(X, Y; D_1, D_2) &= \lim_{\epsilon \rightarrow 0} C_W^{LB}(D_1, D_2, \epsilon) \\ &= \lim_{\epsilon \rightarrow 0} C_W^{UB}(D_1, D_2, \epsilon) \end{aligned}$$

The proof of this Lemma follows directly by observing that, if there exists a joint density  $P(\hat{X}, \hat{Y}|X, Y)$  that achieves  $R_{X,Y}(D_1, D_2)$ , then both the above limits converge to  $C_W^{UB}(D_1, D_2, 0)$ , which is same as the information theoretic characterization in (18). ■

## APPENDIX B

### PROOF OF COROLLARY 1

*Proof:* Our objective is to prove that the optimizing distribution in Theorem 1 satisfies (27). We begin with an auxiliary property of the RD-optimal conditional distribution  $P(\hat{X}^*, \hat{Y}^*|X, Y)$ . Recall that the RD optimal conditional distribution minimizes  $I(X, Y; \hat{X}^*, \hat{Y}^*)$  over all joint distributions that satisfy the distortion constraints,  $E(d_X(X, \hat{X}^*)) \leq D_1$  and  $E(d_Y(Y, \hat{Y}^*)) \leq D_2$ . Hence, it follows using standard arguments [11] that, for every distortion pair  $(D_1, D_2)$ , the RD-optimal conditional distribution,  $P(\hat{X}^*, \hat{Y}^*|X, Y)$ , also minimizes the following Lagrangian for some positive constants  $\mu_1, \mu_2$  and positive valued function  $\lambda(x, y)$  defined for all  $x \in \mathcal{X}, y \in \mathcal{Y}$ :

$$\begin{aligned} L &= I(X, Y; \hat{X}^*, \hat{Y}^*) \\ &\quad - \mu_1 E(d_X(X, \hat{X}^*)) - \mu_2 E(d_Y(Y, \hat{Y}^*)) \\ &\quad - \int \lambda(x, y) dx dy \int d\hat{x}^* d\hat{y}^* P(\hat{x}^*, \hat{y}^*|x, y) \end{aligned} \quad (82)$$

Upon differentiating the above Lagrangian with respect to  $P(\hat{x}^*, \hat{y}^*|x, y)$  and setting it to zero leads to the necessary conditions for optimality of the joint distribution. Some routine steps and simplifications lead to the conclusion that the joint distribution  $P(X, Y, \hat{X}^*, \hat{Y}^*)$  must satisfy the following:

$$\begin{aligned} P(\hat{X}^*, \hat{Y}^*|X, Y) &= \lambda(X, Y)P(\hat{X}^*, \hat{Y}^*) \\ &\quad \times \exp(d_X(X, \hat{X}^*)) \exp(d_Y(Y, \hat{Y}^*)) \end{aligned} \quad (83)$$

It follows that the conditional density  $P(X, Y|\hat{X}^*, \hat{Y}^*)$  satisfies:

$$P(X, Y|\hat{X}^*, \hat{Y}^*) = \phi(X, Y) \exp(d_X(X, \hat{X}^*)) \exp(d_Y(Y, \hat{Y}^*)) \quad (84)$$

where  $\phi(X, Y) = \lambda(X, Y)P(X, Y)$ .

Next recall that the optimizing distribution in Theorem 1 satisfies the following two Markov conditions:

$$\begin{aligned} (X, Y) &\leftrightarrow (\hat{X}^*, \hat{Y}^*) \leftrightarrow U \\ \hat{X}^* &\leftrightarrow U \leftrightarrow \hat{Y}^* \end{aligned} \quad (85)$$

Hence the optimizing joint distribution can be rewritten as:

$$\begin{aligned} P(X, Y, \hat{X}^*, \hat{Y}^*, U) &= P(U)P(\hat{X}^*|U)P(\hat{Y}^*|U) \\ &\quad \times P(X, Y|\hat{X}^*, \hat{Y}^*) \\ &= \phi_1(X, \hat{X}^*, U)\phi_2(Y, \hat{Y}^*, U)\phi(X, Y) \end{aligned} \quad (86)$$

where  $\phi_1(X, \hat{X}^*, U) = P(U)P(\hat{X}^*|U) \exp(d_X(X, \hat{X}^*))$  and  $\phi_2(Y, \hat{Y}^*, U) = P(\hat{Y}^*|U) \exp(d_Y(Y, \hat{Y}^*))$ . Hence, it follows that:

$$P(\hat{X}^*, \hat{Y}^*, U|X, Y) = \phi_1(X, \hat{X}^*, U)\phi_2(Y, \hat{Y}^*, U)\lambda(X, Y) \quad (87)$$

which implies that the Markov conditions in (27) must be satisfied, proving the Lemma. ■

## APPENDIX C

### SHANNON LOWER BOUND FOR VECTORS

In this appendix, we review some of the definitions and results pertinent to Shannon lower bounds for vectors of random variables. We refer to [11] (section 4.3.1) for further details on Shannon lower bound and its properties.

Let  $\mathbf{X}$  be an  $n$ -dimensional random variable distributed according to  $p(\mathbf{X})$ , and let  $\rho_i(x_i, \hat{x}_i) \forall i \in \{1, \dots, N\}$  be any well defined difference distortion measures, i.e.,  $\rho_i(x, \hat{x}) = \rho_i(x - \hat{x})$ . Let  $R_{\mathbf{X}}(\mathbf{D})$  be the rate-distortion function of  $\mathbf{X}$ , with respect to the given distortion measures, i.e.:

$$R_{\mathbf{X}}(\mathbf{D}) = \inf_{P(\hat{\mathbf{X}}|\mathbf{X}):E[\rho_i(x_i, \hat{x}_i)] \leq D_i \forall i} I(\mathbf{X}, \hat{\mathbf{X}}) \quad (88)$$

Then the Shannon lower bound to  $R_{\mathbf{X}}(\mathbf{D})$ , denoted by  $R_{\mathbf{X}}^L(\mathbf{D})$ , is given by:

$$\begin{aligned} R_{\mathbf{X}}^L(\mathbf{D}) &= H(\mathbf{X}) \\ &\quad - \sup_{s_1, \dots, s_n < 0} \sum_{i=1}^N \left\{ s_i D_i - \log \int e^{s_i \rho_i(z_i)} dz_i \right\} \\ &= H(\mathbf{X}) - \sum_{i=1}^N \max_{g_i \in G_i(D_i)} H(g_i) \end{aligned} \quad (89)$$

where,  $G_i(D_i)$  denotes the set of all joint distributions such that:

$$\int \rho_i(z_i) g(z_i) dz_i \leq D_i \quad (90)$$

The above derivation is a direct extension of the derivation in Section 4.3.1 in [11], to vectors of random variables. It is easy to verify that the distribution  $g_i$  that achieves the maximum in (89) is given by:

$$g_i(z) = \frac{e^{s_i \rho(z)}}{\int e^{s_i \rho(z)} dz} \quad (91)$$

where,  $s_i$  is such that:

$$\int g_i(z) \rho_i(z) dz = D_i \quad (92)$$

Shannon showed that  $R_{\mathbf{X}}^L(\mathbf{D}) \leq R_{\mathbf{X}}(\mathbf{D})$  always holds (see [11] for details). The following lemma states the necessary and sufficient conditions for  $R_{\mathbf{X}}^L(\mathbf{D}) = R_{\mathbf{X}}(\mathbf{D})$ .

*Lemma 5:*  $R_{\mathbf{X}}^L(\mathbf{D}) = R_{\mathbf{X}}(\mathbf{D})$  iff the distribution of  $\mathbf{X}$  can be expressed as:

$$p(\mathbf{x}) = \int q(\hat{\mathbf{x}}) \prod_i^n g_i(x_i - \hat{x}_i) d\hat{\mathbf{x}} \quad (93)$$

i.e.,  $\mathbf{X}$  can be expressed as the sum of two statistically independent random vectors,  $\hat{\mathbf{X}}$  and  $\mathbf{Z}$ , where  $\mathbf{Z}$  is distributed according to:

$$g(\mathbf{Z}) = \prod_i^n g_i(z_i) \quad (94)$$

where  $g_i(z)$  is given by (91).

*Proof:* Direct extension of [11, Th. 4.3.1]. ■

It follows from the above Lemma that, if Shannon lower bound is tight,  $\hat{\mathbf{X}}$  is the RD-optimal reconstruction and the RD-optimal backward channel from  $\hat{\mathbf{X}}$  to  $\mathbf{X}$  is additive and can be written as:

$$\mathbf{X} = \hat{\mathbf{X}} + \mathbf{Z} \quad (95)$$

where  $\mathbf{Z} \sim \prod_i^n g_i(Z_i)$ . Therefore, if Shannon lower bound is tight, the components of  $\mathbf{X}$  are independent given  $\hat{\mathbf{X}}$ , i.e., the joint density  $p(\mathbf{X}, \hat{\mathbf{X}})$  is of the form:

$$p(\mathbf{X}, \hat{\mathbf{X}}) = q(\hat{\mathbf{X}}) \prod_i^n p_i(X_i|\hat{X}_i). \quad (96)$$

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