

Generalized Vector Quantization: Jointly Optimal Quantization and Estimation ¹

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Given a pair of random vectors \mathbf{X}, \mathbf{Y} , we study the problem of finding an efficient or optimal estimator of \mathbf{Y} given \mathbf{X} when the range of the estimator is constrained to be a finite set of values. A *generalized vector quantizer* (GVQ), with input dimension k , output dimension m , and size N maps input $\mathbf{X} \in \mathcal{R}^k$, to output $V(\mathbf{X}) \in \mathcal{R}^m$. The output $V(\mathbf{X})$ is constrained to be one of the *estimation codevectors* in the *codebook*, $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_N\}$. The performance of the GVQ is measured by the average distortion, $D = E[d(\mathbf{Y}, V(\mathbf{X}))]$ for a suitable output-space distortion measure $d(\cdot, \cdot)$. A GVQ reduces to a conventional vector quantizer in the special case where $\mathbf{X} = \mathbf{Y}$. The GVQ problem has been approached in the information theory literature from many different standpoints. In particular, it appears in the context of noisy source coding, which is the special case where we quantize \mathbf{X} , the observable, noisy version of a source, \mathbf{Y} .

A GVQ partitions the input space \mathcal{R}^k into N decision regions or *cells*. Each cell is mapped by the GVQ to a particular codevector. In principle, a GVQ is fully characterized by specifying (a) the input space partition and (b) the codebook. Correspondingly, one can view the GVQ operation as the composition of two operations, an *encoder*, \mathcal{E} , which assigns an index i to each input vector \mathbf{X} , and a *decoder*, \mathcal{D} , which is a table-lookup operation that generates \mathbf{y}_i , given i . Thus, \mathcal{E} is a classifier whose performance measure is the distortion in \mathbf{Y} induced by the classification, and \mathcal{D} is the conditional estimator of \mathbf{Y} , given the classification index assigned by \mathcal{E} . We summarize the necessary conditions and properties of the optimal GVQ. However, the optimal encoder has, in general, unmanageable complexity since its partition regions may be neither convex nor connected. We propose therefore, to constrain the complexity of the encoder, \mathcal{E} by restricting its structure. Finding the optimal GVQ subject to the structural constraint is a hard optimization problem and to address it, we apply ideas from statistical physics. Although the approach we propose is extendible to a variety of structures, we restrict our derivation to the specific structure of the *multiple prototype classifier* and we refer to such a GVQ system as the multiple-prototype generalized vector quantizer (MP-GVQ). In MP-GVQ, a codevector, \mathbf{y}_j owns M_j prototypes, $\{\mathbf{x}_{j1}, \mathbf{x}_{j2}, \dots, \mathbf{x}_{jM_j}\}$. The encoding rule finds the nearest prototype to the input \mathbf{X} and maps it to the estimation vector associated with that prototype. Thus, the encoder partition region R_j is the union of M_j nearest neighbor Voronoi cells.

The MP-GVQ design problem is to *jointly* optimize the prototypes $\{\mathbf{x}_{jk}\}$ and codevectors $\{\mathbf{y}_j\}$ to minimize the distortion, D . The problem cannot be directly solved with a vari-

ant of Lloyd's algorithm nor by a gradient descent approach, due to the discrete nature of the classifier partition. We tackle the problem by introducing a probabilistic framework for the encoding rule where, for a given input, a probability distribution is assigned to the set of prototypes and the estimation vector assigned to the input is determined by the class index of the randomly chosen prototype. The degree of randomness is measured by the Shannon entropy. Randomization of the nearest-neighbor partition subject to a constraint on the encoder entropy results in the Gibbs distribution for the encoding rule. The Lagrange parameter, γ controls the degree of randomness, and as $\gamma \rightarrow \infty$, the encoding rule approaches the (non-random) nearest-neighbor rule and the entropy goes to zero. Furthermore, this Lagrangian framework is extended to re-formulate the entire MP-GVQ problem as a minimization of the expected distortion, D subject to an entropy constraint. The corresponding Lagrange multiplier, β is inversely related to the temperature in the physical analogy, as explained below.

The method consists of starting with a highly random encoder (large value of the entropy constraint) and gradually reducing the entropy while solving the optimization at each level. At the limit of zero entropy, we obtain a deterministic solution *satisfying the structural constraint and minimizing the output distortion*.

This is an annealing process corresponding to the physical analogy where a system whose energy is the output distortion and whose temperature is inversely related to the Lagrange multiplier, β , is gradually cooled down to zero temperature. This analogy also explains the ability of the method to avoid many local minima that riddle the distortion surface. The physical analogy is taken a step further by observing that the system undergoes phase transitions in the sequence of solutions obtained for decreasing values of entropy. These transitions correspond to an increase in the effective size of the model (the number of distinct codevectors found in the solution for each entropy value). We provide a result yielding the critical temperature (at which a set of codevectors "split" into a larger set) as a function of the covariances and cross-covariances of \mathbf{X} and \mathbf{Y} in the respective clusters. The result extends the original results for phase transitions of deterministic annealing process previously studied for conventional vector quantizer design.

We demonstrate the usefulness of our MP-GVQ design procedure for a variety of examples from the source coding literature.

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